

The Space of Kähler metrics

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Abstract

Donaldson conjectured[14] that the space of Kähler metrics is geodesic convex by smooth geodesic and that it is a metric space. Following Donaldson's program, we verify the second part of Donaldson's conjecture completely and verify his first part partially. We also prove that the constant scalar curvature metric is unique in **each** Kähler class if the first Chern class is either strictly negative or 0. Furthermore, if $C_1 \leq 0$, the constant scalar curvature metric realizes the global minimum of Mabuchi energy functional; thus it provides a new obstruction for the existence of constant curvature metric: if the infimum of Mabuchi energy (taken over all metrics in a fixed Kähler class) isn't bounded from below, then there doesn't exist a constant curvature metric. This extends the work of Mabuchi and Bando[3]: they showed that Mabuchi energy bounded from below is a necessary condition for the existence of Kähler-Einstein metrics in the first Chern class.

1 Introduction to the problem

1.1 Brief introduction to the classical problems in Kähler geometry

Let V be a Kähler manifold. E. Calabi conjectured in 1954 that any $(1,1)$ form which represents $C_1(V)$ (the first Chern class) is the Ricci form of some Kähler metrics on V . Yau [44], in 1978, proved this Calabi's conjecture. Around the same time, Aubin [1] and Yau proved independently the existence of a Kähler-Einstein metric on a Kähler manifold with negative first Chern class (also a conjecture of E. Calabi). G. Tian [36], in 1987, proved the existence of Kähler-Einstein metric in a canonical Kähler class on complex surfaces if the first Chern class is positive and the group of automorphism is reductive. For further references on this subject, see [37] and [38]. An important conjecture by Yau [45] relates the existence of Kähler-Einstein metrics to the stability in the sense of Hilbert schemes and Geometric invariant theory.

Kähler-Einstein metrics could be treated as a special kind of extremal Kähler metrics. The question of extremal Kähler metric was first raised by E. Calabi in his paper[9] : he considered L^2 norm of curvature as a functional from a given Kähler class; a critical point of this functional is called an "extremal Kähler metric." He showed that any extremal Kähler metric must be symmetric under a maximal compact subgroup of the holomorphic transformation group. Using this structure theorem of Calabi, Marc Levine [28] was able to construct a Kähler surface on which there is no extremal Kähler metric. In 1992, D. Burns and P. de Bartolomeis [8] also produced an example of non-existence of extremal Kähler metric; their example suggests some new obstruction for the existence of extremal metrics which is related to some borderline semi-stability of hermitian vector bundle. LeBrun[25] also demonstrated that the existence of critical Kähler metrics might be tied up with the stability of corresponding vector bundles. Donaldson[14] thought that Yau's conjecture [45] should be extend over to the general extremal Kähler metrics. For further references in the subject of extremal metrics, please see [24], [25] [17] and references therein.

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Futaki[16] in 1983 introduced an analytic invariant for any Kähler manifold with positive first Chern class. The vanishing of this invariant is a necessary condition for the existence of a Kähler-Einstein metric on the manifold. Later Futaki and Calabi [10] generalized the invariant to any compact Kähler class. This generalized Futaki invariant, i.e., Calabi-Futaki invariant, is an analytic obstruction to the existence of constant scalar curvature metric in a Kähler manifold. In the same paper, Calabi also shows that constant scalar curvature metric and extremal Kähler metric with non-constant scalar curvature do not co-exist in a single Kähler class.

For the uniqueness, the known results are as follows: 1) in 1950s, E. Calabi showed the uniqueness of Kähler-Einstein metrics if $C_1 \leq 0$. 2) in 1987, Mabuchi and S. Bando [3] showed the uniqueness of Kähler - Einstein metrics up to holomorphic transformation if the first Chern class is positive. Recently, Tian and X.H. Zhu [41] proved the uniqueness of Kähler-Ricci Soliton with respect to a fixed holomorphic vector field on any Kähler manifolds with positive first Chern class. Although very little was known about the uniqueness of general extremal Kähler metrics, most experts in Kähler geometry expect that the extremal Kähler metric is unique in each Kähler class up to holomorphic transformation. In [42] (also see [43] for further references), we demonstrated two degenerate extremal Kähler metrics in the same Kähler class with different energy levels and different symmetry groups: one example is due to Calabi, the other is due to the author. To my knowledge, it appears that this is the only non-uniqueness example known today.

Main results. Mabuchi ([35])* in 1987 defined a Riemannian metric on the space of Kähler metrics, under which it becomes (formally) a non-positive curved infinite dimensional symmetric space. Apparently unaware of Mabuchi's work, Semmes [32] and Donaldson [14] re-discover this same metric again from different angles. In [32], Semmes S. first pointed out that the geodesic equation is a homogeneous complex Monge-Ampere equation on a manifold of one dimension higher. In [14], Donaldson further conjectured that the space is geodesically convex and it is a genuine metric space. We prove that it is at least convex by $C^{1,1}$ geodesics, and from which we conclude that the space is indeed a metric space, thus verifying the second part of Donaldson's conjecture. Moreover, this $C^{1,1}$ geodesic realizes the absolute minimum of length over all possible paths connecting the end points; thus the metric aforementioned is a genuine one. Using these results, we are able to show that the constant curvature metric is unique in each Kähler class if $C_1 < 0$ or $C_1 = 0$. Furthermore, if $C_1 \leq 0$, we show that constant scalar metric (if exists) realizes the global minimum of Mabuchi energy, which gives an affirmative answer to a question raised by Gang Tian [40] in this special case. This last statement also extends the work of Mabuchi and Bando[3]: they showed that Mabuchi energy bounded from below is a necessary condition for the existence of Kähler-Einstein metrics in the first Chern class. In the light of Tian's work[38] in which he shows that in Kähler manifold with positive first Chern class and no non-trivial holomorphic fields, the Kähler-Einstein metric exists if and only if the Mabuchi functional is proper (he actually uses an equivalent functional instead of the Mabuchi functional)[†]. One would like to ask: is this still true for constant scalar curvature metrics[‡]?

Organization: In section 2, we first summarize the different approaches taken by Mabuchi, Semmes and Donaldson independently in the space of Kähler metrics; then we introduce this Riemannian metric on this infinite dimensional space and prove that it has non-positive sectional curvature in the formal sense. Then we introduce Donaldson's two conjectures and reduce the 1st conjecture to the existence problem for the complex homogeneous Monge-Ampere equation with Drichelet boundary data. Readers are alerted that material in section 2.3-2.5 is essentially a re-presentation of Donaldson's work [14], included here for the convenience of readers. In section 3, we prove that this geodesic(CHMA) equation always has a $C^{1,1}$ solution. In section 4, we prove that a continuous

*Around the same time with Mabuchi's work, Bourguignon J. P. has worked on something similar in a related subject [6].

[†]Here we mean the mixed second derivatives is uniformly bounded. See theorem 3 in section 3 for details.

[‡]The sufficient part of this result was proved in [12].

[§]Tian inform us that he[39] has conjectured that constant scalar curvature metrics exist if and only if Mabuchi functional is proper.

solution to the geodesic(CHMA) equation in some appropriate weak sense is unique. In section 5, we show that the geodesic distance defined by the length of $C^{1,1}$ geodesic satisfies the triangular inequality. Using this, we prove the space of Kähler metrics is a metric space. In section 6 we show that extremal Kähler metric is unique in each Kähler class if either $C_1(V) < 0$ or $C_1(V) = 0$.

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2 Space of Kähler metrics

2.1 Mabuchi and S. Semmes' Ideas

Shortly after introducing the now famous Mabuchi functional, Mabuchi [35] set out and defined a Riemannian metric in the space of Kähler metrics. Besides showing formally it is a locally symmetric space with non-positive sectional curvature, he also pointed out that the Mabuchi energy is formally convex in this infinite dimensional space (in the sense that the Hessian is semi-positive definite). Perhaps, this is his original motivation for introducing such a metrics. Unaware of Mabuchi's work, in a remarkable paper [32], S. Semmes studied the geometry of solution of complex Homogeneous Monge-Ampere equation (CHMA). He observed that in some special domain $\Omega \times D$ where Ω is n-dimensional domain in C^n and D is a domain in complex plane, the solution to CHMA is some sort of geodesic equation if the data is rotationally symmetric when restricted to D . He then considered the space of pluri-subharmonic functions in Ω and defined a Riemannian metric in this space according to this geodesic equation. It turns out that this space becomes non-positively curved (locally) symmetric space in some formal sense. He also went on to study the variational problem of finding a geodesic. It seems that he is mainly motivated from providing a proper geometric meaning to solution of CHMA with right domain setting. Unlike real homogeneous Monge-Ampere equation (RHMA) whose solution always has proper geometric meaning, solution of a CHMA equation doesn't have a preferred geometric interpretation. Without a proper geometry interpretation, it is very hard to work on this subject. Of course, great progress has been made since the famous work of L. Caffarelli, L. Nirenberg and J. Spruck[22] and later their joint work with J. Kohn [21]. For instance, Lempert L. [27], E. Bedford and B.A. Taylor [4]; leong P.[26] and important work of Krylov [20] and Evans [23] ...etc.. This is by no means a complete list of papers in complex Monge-Ampere equation since the author is quite new to this important field. For a complete and updated references, please see S. Kolodziej [19]. Donaldson's recent work certainly makes Mabuchi and Semmes's original work all the more remarkable.

2.2 Brief summary of Donaldson's theory on space of Kähler metrics

Motivated from complete different reasons, S. K. Donaldson [14] re-discovered this metric. More importantly, he outlined a strategy in [14] to relate this geometry of infinite dimensional space to the existence problems in Kähler geometry. In particular, he explains how one can uses this extra structure in the infinite dimensional space to solve the problems of the existence and uniqueness of extremal Kähler metrics. In general, the later are intractable problems from traditional means. He regards the space of Kähler metrics in a fixed Kähler class as an infinite dimensional symplectic manifold with the automorphism group $SDiff(V)$ (symplectic diffeomorphism group of V into itself). In [13], he pointed out that scalar curvature is the moment map μ from this infinite dimension

symplectic manifold to the dual space of the Lie algebra of its automorphism group[¶]. Thus, to find an extremal Kähler metric in a fixed Kähler class in classical Kähler geometry could be re-interpreted as to find a pre-image of 0 of the moment map μ in this symplectic setting. This acute observation sheds new light into the otherwise intractable problem of the existence of extremal Kähler metrics in a Kähler manifold; at least conceptually, the picture looks much clear. He then proposed several conjectures whose ultimate resolution will lead to a better understanding of extremal Kähler metric, and for that matter, better understanding of Kähler geometry as well. The most fundamental one among his conjectures is the so called geodesic conjecture: any two Kähler metrics in the same class is connected by a smooth geodesic. A second conjecture by him is that this space of Kähler metric is a metric space under this metric. If the geodesic conjecture is true, this second conjecture will be a direct consequence (since this space of Kähler metrics in a fixed Kähler class is non-positively curved in the formal sense.). He went on to show that the uniqueness of extremal Kähler metric is a consequence of this geodesic conjecture as well.

2.3 Riemannian metrics in the infinite dimensional space.

Now we introduce this metric here. Readers are referred to Mabuchi, S. Semmes and Donaldson's original writing for details. Consider the space of Kähler potentials in a fixed Kähler class as:

$$\mathcal{H} = \{\varphi \in C^\infty(V) : \omega_\varphi = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi > 0 \text{ on } V\}.$$

Clearly, the tangent space $T\mathcal{H}$ is $C^\infty(V)$. Each Kähler potential $\phi \in \mathcal{H}$ defines a measure $d\mu_\phi = \frac{1}{n!}\omega_\phi^n$. Now we define a Riemannian metric on the infinite dimensional manifold \mathcal{H} using the L^2 norm provided by these measures. A tangent vector in \mathcal{H} is just a function in V . For any vector $\psi \in T_\varphi\mathcal{H}$, we define the length of this vector as

$$\|\psi\|_\varphi^2 = \int_V \psi^2 d\mu_\varphi.$$

For a path $\varphi(t) \in \mathcal{H}(0 \leq t \leq 1)$, the length is given by

$$\int_0^1 \sqrt{\int_V \varphi(t)'^2 d\mu_{\varphi(t)}} dt$$

and the geodesic equation is

$$\varphi(t)'' - \frac{1}{2}|\nabla\varphi'(t)|_{\varphi(t)}^2 = 0, \quad (1)$$

where the derivative and norm in the 2nd term of the left hand side are taken with respect to the metric $\omega_{\varphi(t)}$.

This geodesic equation shows us how to define a connection on the tangent bundle of \mathcal{H} . The notation is simplest if one thinks of such a connection as a way of differentiating vector fields along paths. Thus, if $\phi(t)$ is any path in \mathcal{H} and $\psi(t)$ is a field of tangent vectors along the path (that is, a function on $V \times [0, 1]$), we define the covariant derivative along the path to be

$$D_t\psi = \frac{\partial\psi}{\partial t} - \frac{1}{2}(\nabla\psi, \nabla\phi')_\phi.$$

This connection is torsion-free because in the canonical “co-ordinate chart”, which represents \mathcal{H} as an open subset of $C^\infty(V)$, the “Christoffel symbol”

$$\Gamma : C^\infty(V) \times C^\infty(V) \rightarrow C^\infty(V)$$

[¶]The *moment map* point of view here was also observed by A. Fujiki[15].

at ϕ is just

$$\Gamma(\psi_1, \psi_2) = -\frac{1}{2}(\nabla\psi_1, \nabla\psi_2)_\phi$$

which is symmetric in ψ_1, ψ_2 . The connection is metric-compatible because

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\psi\|_\phi^2 &= \frac{d}{dt} \int_V \psi^2 d\mu_\phi \\ &= \int_V \frac{\partial \psi}{\partial t} \psi + \frac{1}{2} \psi^2 \Delta(\phi') d\mu_\phi \\ &= \int_V \frac{\partial \psi}{\partial t} \psi - \frac{1}{4} (\nabla(\psi^2), \nabla\phi')_\phi d\mu_\phi \\ &= \int_V \left(\frac{\partial \psi}{\partial t} - \frac{1}{2} (\nabla\psi, \nabla\phi')_\phi \right) \psi d\mu_\phi \\ &= \langle D_t \psi, \psi \rangle. \end{aligned}$$

Here Δ is complex Laplacian operator. The main theorem proved in [35] (and later reproved in [32] and [14]) is:

Theorem A *The Riemannian manifold \mathcal{H} is an infinite dimensional symmetric space; it admits a Levi-Civita connection whose curvature is covariant constant. At a point $\phi \in \mathcal{H}$ the curvature is given by*

$$R_\phi(\delta_1\phi, \delta_2\phi)\delta_3\phi = -\frac{1}{4}\{\{\delta_1\phi, \delta_2\phi\}_\phi, \delta_3\phi\}_\phi,$$

where $\{ , \}_\phi$ is the Poisson bracket on $C^\infty(V)$ of the symplectic form ω_ϕ ; and $\delta_1\phi, \delta_2\phi \in T_\phi\mathcal{H}$.

(Recall that in infinite dimensions the usual argument gives the uniqueness of a Levi-Civita [i.e. torsion-free, metric-compatible] connection, but not the existence in general.) The formula for the curvature of \mathcal{H} entails that the sectional curvature is non-positive, given by

$$K_\phi(\delta_1\phi, \delta_2\phi) = -\frac{1}{4}\|\{\delta_1\phi, \delta_2\phi\}_\phi\|_\phi^2.$$

Different proofs of this theorem have been appeared in [35], [32] and [14]. We will skip the proof here, interested readers are referred to these papers if they are interested in the proof.

The expression for the curvature tensor in terms of Poisson brackets shows that R is invariant under the action of the symplectic-morphism group. Since the connection on $T\mathcal{H}$ is induced from an SDiff-connection, it follows that R is covariant constant, and hence \mathcal{H} is indeed an infinite-dimensional symmetric space.

2.4 Splitting of \mathcal{H}

There is obviously a decomposition of the tangent space:

$$T_\phi\mathcal{H} = \{\psi : \int_V \psi d\mu_\phi = 0\} \oplus \mathbf{R}.$$

We claim that this corresponds to a Riemannian decomposition

$$\mathcal{H} = \mathcal{H}_0 \times \mathbf{R}.$$

We are interested to see this Riemannian splitting more explicitly, partly because we see the appearance of a functional I on the space of Kähler potentials, which is well-known in the literature, see [2], [38] for example. The decomposition of tangent space of \mathcal{H} gives a 1-form α on \mathcal{H} with

$$\alpha_\phi(\psi) = \int_V \psi d\mu_\phi,$$

and it is straightforward to verify that this 1-form is *closed*. Indeed

$$(d\alpha)_\phi(\psi, \tilde{\psi}) = \int_V (\tilde{\psi}\Delta\psi - \psi\Delta\tilde{\psi}) = 0.$$

This means that there is a function $I : \mathcal{H} \rightarrow \mathbf{R}$ with $I(0) = 0$ and $dI = \alpha$, and it is this function which gives rise to the corresponding Riemannian decomposition. We call a Kähler potential ϕ *normalized* if $I(\phi) = 0$. Then any Kähler metric has a unique normalized potential, and the restriction of our metric on \mathcal{H} to $I^{-1}(0)$ endows the space \mathcal{H}_0 of Kähler metrics with a Riemannian structure; this is independent of the choice of base point ω_0 and clearly makes \mathcal{H}_0 into a symmetric space. The functional I can be written more explicitly by integrating α along lines in \mathcal{H} to give the formula

$$I(\phi) = \sum_{p=0}^n \frac{1}{(p+1)!(n-p)!} \int_V \omega_0^{n-p} (\partial \bar{\partial} \phi)^p \phi.$$

2.5 Donaldson' Conjectures

We will now study the geodesic equation in \mathcal{H} in more detail, and interpret the solutions geometrically. Suppose ϕ_t , $t \in [0, 1]$, is a path in \mathcal{H} . We can view this as a function on $V \times [0, 1]$ and in turn as a function on $V \times [0, 1] \times S^1$, with trivial dependence on the S^1 factor; that is, we define

$$\Phi(v, t, e^{is}) = \phi_t(v).$$

We regard the cylinder $\mathbf{R} = [0, 1] \times S^1$ as a Riemann surface with boundary in the standard way—so $t + is$ is a local complex co-ordinate. Let Ω_0 be the pull-back of ω_0 to $V \times \mathbf{R}$ under the projection map and put $\Omega_\Phi = \Omega_0 + \partial \bar{\partial} \Phi$, a $(1, 1)$ -form on $V \times \mathbf{R}$. Then we have:

Proposition 1 *The path ϕ_t satisfies the geodesic equation (1) if and only if $\Omega_\Phi^{n+1} = 0$ on $V \times \mathbf{R}$.*

Proof: Denote the metric defined by ω_0, ω_ϕ as g, g' . Then

$$\frac{1}{n!} \omega_\phi^n = \det g'; \quad \frac{1}{n!} \omega_0^n = \det g.$$

Then geodesic equation is equivalent to the following (if $\det g' \neq 0$)

$$(\phi'' - \frac{1}{2} |\nabla \phi'|_{g'}^2) \det g' = 0.$$

The last equation is equivalent to

$$\det \begin{pmatrix} g' & \begin{pmatrix} \frac{\partial \phi'}{\partial z_1} \\ \frac{\partial \phi'}{\partial z_2} \\ \vdots \\ \frac{\partial \phi'}{\partial z_n} \end{pmatrix} \\ \begin{pmatrix} \frac{\partial \phi'}{\partial \bar{z}_1} & \frac{\partial \phi'}{\partial \bar{z}_2} & \dots & \frac{\partial \phi'}{\partial \bar{z}_n} \end{pmatrix} & \phi'' \end{pmatrix} = 0.$$

Let $w = t + \sqrt{-1}s$, then $t = \text{Re}(w)$. The above equation could be re-written as

$$\det \begin{pmatrix} (g + \frac{\partial^2 \phi}{\partial z_\alpha \partial \bar{z}_\beta})_{n \times n} & \begin{pmatrix} \frac{\partial^2 \phi}{\partial z_1 \partial \bar{w}} \\ \frac{\partial^2 \phi}{\partial z_2 \partial \bar{w}} \\ \vdots \\ \frac{\partial^2 \phi}{\partial z_n \partial \bar{w}} \end{pmatrix} \\ \begin{pmatrix} \frac{\partial^2 \phi}{\partial \bar{z}_1 \partial w} & \frac{\partial^2 \phi}{\partial \bar{z}_2 \partial w} & \dots & \frac{\partial^2 \phi}{\partial \bar{z}_n \partial w} \end{pmatrix} & \frac{\partial^2 \phi}{\partial w \partial \bar{w}} \end{pmatrix} = 0.$$

This is just $\Omega_\Phi^{n+1} = 0$. The proposition is then proved.

QED.

Given boundary data —a real value function $\rho \in C^\infty(\partial(V \times \mathbf{R}))$, we consider the set of functions Φ on $V \times \mathbf{R}$ which agree with ρ on the boundary. Then we define the variation of I_ρ on this set by

$$\delta I_\rho = \frac{1}{(n+1)!} \int_{V \times \mathbf{R}} \delta \Phi \Omega_\Phi^{n+1},$$

where the variation $\delta \Phi$ vanishes on the boundary by hypothesis. This boundary condition means that we can show easily that this formula defines a functional I_ρ . To prove this, one only need to show that the second derivatives of I_ρ with respect to two infinitesimal variation $\delta_1 \Phi$ and $\delta_2 \Phi$ is symmetric. The second derivatives is:

$$\frac{1}{2} \cdot \frac{1}{(n+1)!} \int_V \delta_1 \Phi \triangle \delta_2 \Phi \Omega_\Phi^{n+1}$$

which is clearly symmetric on $\delta_1 \Phi$ and $\delta_2 \Phi$. Here \triangle is the Laplacian operator of Ω_Φ on $V \times \mathbf{R}$.

This functional I_ρ reduces to the energy functional on paths, by an integration by parts, in the case when \mathbf{R} is the cylinder and we restrict to S^1 -invariant data. Suppose $\phi(t)$ ($0 \leq t \leq 1$) is a path in \mathcal{H} , and $\delta \phi$ represents the infinitesimal variation of ϕ while keep value of ϕ fixed when $t = 0, 1$. Thus, the variation of I_ρ in $\delta \phi$ direction is (follow notations in the proof of previous proposition):

$$\delta I_\rho = \frac{1}{(n+1)!} \int_{V \times R} \delta \phi \Omega_\Phi^{n+1} = \frac{1}{(n+1)!} \int_{t=0}^1 \int_V \delta \phi (\phi'' - \frac{1}{2} |\nabla \phi'|_{g'}^2) \det g' dt.$$

On the other hand, the variation of energy functional along this path is:

$$\delta E = \int_{t=0}^1 \int_V \delta \phi (\phi'' - \frac{1}{2} |\nabla \phi'|_{g'}^2) \det g' dt$$

where $E = \int_{t=0}^1 \int_V \phi'(t)^2 \det g' dt$. Thus, in case when \mathbf{R} is the cylinder and we restrict to S^1 -invariant data, I_ρ equal to the energy functional on the path up to a multiple of constant.

The following is the first conjecture by Donaldson in [14]:

Conjecture 1 (Donaldson) *Let \mathbf{R} be a compact Riemann surface with boundary and $\rho : V \times \partial R \rightarrow \mathbf{R}$ be a function such that $\omega_0 - \sqrt{-1} \bar{\partial} \partial \rho$ is a strictly positive $(1,1)$ form on each slice $V \times \{z\}$ for each fixed $z \in \partial R$. Let \mathcal{S}_ρ be the set of functions Φ on $V \times R$ equal to ρ over the boundary and such that $\omega_0 - \sqrt{-1} \bar{\partial} \partial \Phi$ is strictly positive on every slice $V \times \{w\}$, $w \in R$. Then there is a unique solution of the Monge-Ampere equation $(\Omega_0 - \sqrt{-1} \bar{\partial} \partial \Phi)^{n+1} = 0$ in \mathcal{S}_ρ , and this solution realizes the absolute minimum of the functional I_ρ .*

This question is a version of the Dirichlet problem for the complete degenerate Monge-Ampere equation, a topic around which there is a substantial literature; see [2],[18] for example. Note that regularity questions are very important in this theory, since the equation is not elliptic.

In the case of the geodesic problem, when the functional can be rewritten as the energy of a path; if these infimum are strictly positive, for all choices of fixed, distinct, end points, they make \mathcal{H} into a metric space, in the usual fashion. In this connection, Donaldson proposes the following conjecture (after verifying that it will be satisfied by a smooth geodesic):

Conjecture 2 (Donaldson) *If $\phi \in \mathcal{H}_0$ is normalized and $\tilde{\phi}_t$, $t \in [0, 1]$ is any path from 0 to ϕ in \mathcal{H} then*

$$\int_0^1 \int_V \left(\frac{d\tilde{\phi}}{dt} \right)^2 d\mu_{\tilde{\phi}_t} dt \geq M^{-1} \left(\max \left(\int_{\phi > 0} \phi d\mu_\phi, - \int_{\phi < 0} \phi d\mu_0 \right) \right)^2. \quad (2)$$

The restriction to normalized potentials ϕ is not important since we know that \mathcal{H} splits as a product, and we could immediately write down a corresponding inequality, involving $I(\phi)$, for any $\phi \in \mathcal{H}$. If this conjecture and the geodesic conjecture are proved, then \mathcal{H} is a metric space.

we want to use continuous method to treat this existence problem of geodesic between any two points in \mathcal{H} .

3 Existence of $C^{1,1}$ solution

Let V be a n -dimensional Kähler manifold without boundary, \mathbf{R} be a Riemann surface with boundary. The case we concerned most is when \mathbf{R} is a cylinder. Suppose $g = g_{\alpha\bar{\beta}} dz_\alpha d\bar{z}_\beta$ ($1 \leq \alpha, \beta \leq n$) is a given Kähler metric in V . Then $\tilde{g} = g_{\alpha\bar{\beta}} dz_\alpha d\bar{z}_\beta + dw d\bar{w}$ is a Kähler metric in $V \times \mathbf{R}$, and $\tilde{\varphi} = \varphi - |w|^2$. For convenience, we still denote \tilde{g} as g , and $\tilde{\varphi}$ as φ when there is no confusion arisen. Also, let $z_{n+1} = w$. Then $z = (z_1, z_2, \dots, z_n, z_{n+1})$ is a point in $V \times \mathbf{R}$ and $z' = (z_1, z_2, \dots, z_n)$ is a point in V . Let $\varphi(z) = \varphi(z', w)$ be a function in $V \times \mathbf{R}$ such that $g + \partial_{z'} \bar{\partial}_{z'} \varphi(z', w)$ is a Kähler metric in V for each $w \in \mathbf{R}$. We want to solve the degenerated Monge-Ampere equation:

$$\det \left(g + \frac{\partial^2 \varphi}{\partial z_\alpha \partial \bar{z}_\beta} \right)_{(n+1)(n+1)} = 0 \text{ in } V \times \mathbf{R}; \quad \text{and } \varphi = \varphi_0 \text{ in } \partial(V \times \mathbf{R}). \quad (3)$$

We want to use the continue method to solve this equation. Consider the continuous equation $0 \leq t \leq 1$.

$$\det \left(g + \frac{\partial^2 \varphi}{\partial z_\alpha \partial \bar{z}_\beta} \right) = t \det \left(g + \frac{\partial^2 \varphi_0}{\partial z_\alpha \partial \bar{z}_\beta} \right), \text{ in } V \times \mathbf{R}; \quad \text{and } \varphi = \varphi_0 \text{ in } \partial(V \times \mathbf{R}). \quad (4)$$

Suppose φ_0 is a solution to (4) at $t = 1$ such that $\sum_{\alpha, \beta=1}^{n+1} (g_{\alpha\bar{\beta}} + \frac{\partial^2 \varphi_0}{\partial z_\alpha \partial \bar{z}_\beta}) dz_\alpha d\bar{z}_\beta$ is strictly positive Kähler metric in $V \times \mathbf{R}^\parallel$. Denote $f = \det \left(g + \frac{\partial^2 \varphi_0}{\partial z_\alpha \partial \bar{z}_\beta} \right) (\det g)^{-1} > 0$. Then equation (4) can be re-written in a better form

$$\det \left(g + \frac{\partial^2 \varphi}{\partial z_\alpha \partial \bar{z}_\beta} \right) = t \cdot f \cdot \det(g) \text{ in } V \times \mathbf{R}; \quad \text{and } \varphi = \varphi_0 \text{ in } \partial(V \times \mathbf{R}). \quad (5)$$

Clearly, φ_0 is the unique solution to this equation at $t = 1$. Since the equation is elliptic, this equation can be uniquely solved for t sufficiently closed to 1 (the kernel of linearized operator is zero for any $t > 0$). Let t_0 be such that (5) has a unique smooth solution for every $t \in (t_0, 1]$. We want to show that $t_0 = 0$ in this section. Observe that equation (5) is elliptic for every $t > 0$. Hence, the solution will be as smooth as the boundary value once we show that 2nd derivatives of φ is uniformly bounded. Let h be a super harmonic function on $V \times \mathbf{R}$ with respect to g such that $\Delta_g h + n + 1 = 0$. and $h = \varphi_0$ in $\partial(V \times \mathbf{R})$. Then for any solution of equation (5) for $t < 1$, we have C^0 bound of the solution:

Lemma 1 *If φ is a solution of equation (5) at $0 < t < 1$, then φ has the following a priori C^0 estimate due to maximum principal:*

$$\varphi_0 \leq \varphi \leq h, \quad \text{in } V \times \mathbf{R}.$$

For C^2 estimate, we follow Yau's famous work in Calabi's conjecture. Essentially, we reduce it to a boundary estimate since we have C^0 estimate:

Lemma 2 (Yau) *If φ is a solution of equation (5) at $0 < t < 1$, then φ has the following a priori C^2 estimate:*

$$\begin{aligned} \Delta'(e^{-C\varphi}(n+1+\Delta\varphi)) &\geq e^{-C\varphi}(\Delta \ln f - (n+1)^2 \inf_{i \neq l} (R_{i\bar{i}l\bar{l}})) - Ce^{-C\varphi}(n+1)(n+1+\Delta\varphi) \\ &\quad + (C + \inf_{i \neq l} (R_{i\bar{i}l\bar{l}})) e^{-C\varphi}(n+1+\Delta\varphi)^{1+\frac{1}{n}}(tf)^{-1}. \end{aligned}$$

^{||}By definition, for any $\varphi_0 \in \mathcal{H}$, $\sum_{\alpha, \beta=1}^{n+1} (g_{\alpha\bar{\beta}} + \frac{\partial^2 \varphi_0}{\partial z_\alpha \partial \bar{z}_\beta}) dz_\alpha d\bar{z}_\beta$ is strictly positive Kähler metric in each V -slice $V \times \{w\}$. Let Ψ be a strictly convex function of w which vanishes on $\partial\mathbf{R}$. Then for large enough constants m , $\sum_{\alpha, \beta=1}^{n+1} (g_{\alpha\bar{\beta}} + \frac{\partial^2(\varphi_0 + m\Psi)}{\partial z_\alpha \partial \bar{z}_\beta}) dz_\alpha d\bar{z}_\beta$ is a strictly positive Kähler metric in $V \times \mathbf{R}$.

where $C + \inf_{i \neq l} (R_{i\bar{i}l\bar{l}}) > 1$, Δ is the Laplacian operator with respect to g , while Δ' is the Laplacian operator with respect to $g' = g + \frac{\partial^2 \varphi}{\partial z_\alpha \partial \bar{z}_\beta} dz_\alpha d\bar{z}_\beta$ and $R_{i\bar{i}l\bar{l}}$ is the Riemannian curvature of g .

From the a priori estimate in Lemma 2, either $e^{-C\varphi}(n+1+\Delta\varphi)$ is uniformly bounded in $V \times \mathbf{R}$ or it achieves maximum value at $\partial(V \times \mathbf{R})$. Lemma 1 asserts that φ is uniformly bounded from above and below, then

Corollary 1 *There exists a constant C which depends only on $(V \times \mathbf{R}, g)$ such that*

$$\max_{V \times \mathbf{R}} (n+1+\Delta\varphi) \leq C(1 + \max_{\partial(V \times \mathbf{R})} (n+1+\Delta\varphi)).$$

Theorem 1 *If φ is a solution of equation (5) at $0 < t < 1$, then there exists a constant C which depends only on $(V \times \mathbf{R}, g)$ such that:*

$$\max_{V \times \mathbf{R}} (n+1+\Delta\varphi) \leq C \max_{V \times \mathbf{R}} (|\nabla\varphi|_g^2 + 1). \quad (6)$$

In light of Corollary 1, we only need to prove the inequality (6) on the boundary, i.e.,

$$\max_{\partial(V \times \mathbf{R})} (n+1+\Delta\varphi) \leq C \max_{V \times \mathbf{R}} (|\nabla\varphi|_g^2 + 1).$$

We will prove this inequality in the next subsection.

Theorem 2 *If $\varphi_i (i = 1, 2, \dots)$ are solutions of equation (5) at $0 < t_i < 1$, and the inequality (6) holds uniformly for all these solutions $\{\varphi_i, i \in \mathbf{N}\}$, then there exists a constant C_1 independent of i such that*

$$\max_{V \times \mathbf{R}} (n+1+\Delta\varphi) \leq C \max_{V \times \mathbf{R}} (|\nabla\varphi|_g^2 + 1) < C_1.$$

This is proved via a blowing up argument. We will show this in subsection 3.2.

Remark 1 *By now it is standard estimate of Monge-Ampere equations, that if*

$$\max_{V \times \mathbf{R}} (n+1+\Delta\varphi) \leq C \max_{V \times \mathbf{R}} (|\nabla\varphi|_g^2 + 1) < C_1$$

then equation (5) for t_1, t_2, \dots is a sequence of uniform elliptic equations. The higher derivative of the solution φ_i has a uniform bound as long as $\liminf_{i \rightarrow \infty} t_i > 0$.

Theorem 3 *There exists a $C^{1,1}(V \times \mathbf{R})$ function which solves equation (3) weakly. In other words, for any two points $\varphi_0, \varphi_1 \in \mathcal{H}$, there exists a geodesic path $\varphi(t) : [0, 1] \rightarrow \overline{\mathcal{H}}$ and a uniform constant C such that the following holds:*

$$0 \leq \left(g_{i\bar{j}} + \frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j} \right)_{(n+1)(n+1)} \leq C \left(\tilde{g}_{i\bar{j}} \right)_{(n+1)(n+1)}.$$

Here z_1, z_2, \dots, z_n are local coordinates in V and $t = \text{Re}(z_{n+1})$. And $\tilde{g} = g_{\alpha\bar{\beta}} dz_\alpha d\bar{z}_\beta + dw d\bar{w}$ is a fixed product metric in $V \times \mathbf{R}$.

Following notations in theorem 2, we want to show that $t_0 = \liminf_{i \rightarrow \infty} t_i = 0$. Otherwise, assume $t_0 > 0$. Then equation (5) has a unique smooth solution for $1 \geq t > t_0$. Following from theorem 2, then we have uniform upper bound for $\Delta\varphi + (n+1)$ for all $t_i > t_0 > 0$. Then equation (5) implies that $g'_i = g + \frac{\partial^2 \varphi_i}{\partial z_\alpha \partial \bar{z}_\beta} dz_\alpha d\bar{z}_\beta$ is bounded uniformly from below by a uniform positive constant (this positive low bound approaches 0 when $t \rightarrow 0$). Thus, from equation (5), we obtain uniform higher derivative estimates for solution φ_i . Therefore these solution converge to a regular solution at $t_0 > 0$. Again, since equation (5) at t_0 is an elliptic equation and the kernel of the linearized operator is zero, it can then be solved for any t sufficiently closed to t_0 . But this contradicts to the definition of t_0 . Thus $t_0 = 0$. We can choose a subsequence of $t_i \rightarrow 0$ such that φ_i converge weakly in $C^{1,1}(V \times \mathbf{R})$ where Ω is relative compact subset of $V \times \mathbf{R}$. Again via maximum principal, we can show this limit is unique and define a weak solution of equation (3).

3.1 Boundary estimate

We want to estimate $\triangle\varphi$ at any point in the boundary $\partial(V \times \mathbf{R}) = V \times \partial\mathbf{R}$. Let p be a generic point in $\partial(V \times \mathbf{R})$. Now choose a small neighborhood U of p in $V \times \mathbf{R}$ (this will be a half geodesic ball since $p \in \partial(V \times \mathbf{R})$) and a local coordinate chart such that $g_{\alpha\bar{\beta}}(p) = \delta_{\alpha\bar{\beta}}$ and $p = (z = 0)$

$$\frac{1}{2}\delta_{\alpha\bar{\beta}} \leq g_{\alpha\bar{\beta}}(q) \leq 2\delta_{\alpha\bar{\beta}}, \quad \forall q \in U.$$

Since $\sum_{\alpha,\beta=1}^{n+1} (g_{\alpha\bar{\beta}} + \frac{\partial^2 \varphi_0}{\partial z_\alpha \partial \bar{z}_\beta}) dz_\alpha d\bar{z}_\beta$ is a positive Kähler metric in $V \times \mathbf{R}$, there exists a constant $\epsilon > 0$ such that

$$g_{\alpha\bar{\beta}} + \frac{\partial^2 \varphi_0}{\partial z_\alpha \partial \bar{z}_\beta} > 2\epsilon \cdot g_{\alpha\bar{\beta}}, \quad \text{in } V \times \mathbf{R}.$$

In the neighborhood U of p , we have

$$g_{\alpha\bar{\beta}} + \frac{\partial^2 \varphi_0}{\partial z_\alpha \partial \bar{z}_\beta} > \epsilon \cdot \delta_{\alpha\bar{\beta}} \quad \text{in } V \times \mathbf{R}. \quad (7)$$

We have the trivial estimates in $\partial(V \times \mathbf{R})$:

$$\frac{\partial(\varphi - \varphi_0)}{\partial z_\alpha} = 0, \quad \frac{\partial^2(\varphi - \varphi_0)}{\partial z_\alpha \partial \bar{z}_\beta} = 0, \quad \forall 1 \leq \alpha, \beta \leq n.$$

In order to estimate $\triangle\varphi = \sum_{\alpha,\beta=1}^{n+1} g^{\alpha\bar{\beta}} \frac{\partial^2 \varphi}{\partial z_\alpha \partial \bar{z}_\beta}$ in $\partial(V \times \mathbf{R})$, we only need to estimate $\frac{\partial^2(\varphi - \varphi_0)}{\partial z_\alpha \partial \bar{z}_\beta}$ when either α or β is $n+1$. We will estimate $\frac{\partial^2(\varphi - \varphi_0)}{\partial z_\alpha \partial \bar{z}_{n+1}}$ ($\alpha \leq n$) first, then use equation (5) to derive estimate for $\frac{\partial^2(\varphi - \varphi_0)}{\partial z_{n+1} \partial \bar{z}_{n+1}}$.

Now we set up some conventions:

$$z_\alpha = x_\alpha + \sqrt{-1} y_\alpha, \quad \forall 1 \leq \alpha \leq n; \quad z_{n+1} = x + \sqrt{-1} y$$

where \mathbf{R} near $\partial\mathbf{R}$ is given by $x \geq 0$.

Lemma 3 *There exists a constant C which depends only on $(V \times \mathbf{R}, g)$ such that*

$$|\frac{\partial^2 \varphi}{\partial z_\alpha \partial \bar{z}_{n+1}}(p)| \leq C(\max_{V \times \mathbf{R}} |\nabla \varphi|_g + 1).$$

Proof of theorem 1: At point p , equation (5) reduces to

$$\det(\delta_{\alpha\bar{\beta}} + \frac{\partial^2 \varphi}{\partial z_\alpha \partial \bar{z}_\beta}) = t \cdot f.$$

In other words,

$$\frac{\partial^2 \varphi}{\partial z_{n+1} \partial \bar{z}_{n+1}} = t \cdot f - \frac{\partial^2 \varphi}{\partial z_\alpha \partial \bar{z}_{n+1}} \cdot \frac{\partial^2 \varphi}{\partial \bar{z}_\alpha \partial z_{n+1}}.$$

Lemma 3 then implies that

$$|\frac{\partial^2 \varphi}{\partial z_{n+1} \partial \bar{z}_{n+1}}| \leq C(\max_{V \times \mathbf{R}} |\nabla \varphi|_g^2 + 1).$$

Then,

$$|\triangle\varphi(p)| = |\sum_{\alpha,\beta=1}^{n+1} g^{\alpha\bar{\beta}} \frac{\partial^2 \varphi}{\partial z_\alpha \partial \bar{z}_\beta}(p)| \leq C(\max_{V \times \mathbf{R}} |\nabla \varphi|_g^2 + 1).$$

Since p is a generic point in $\partial(V \times \mathbf{R})$, then theorem 2 holds true. QED.

Let D be any constant linear 1st order operator near the boundary (for instance $D = \pm \frac{\partial}{\partial x_\alpha}, \pm \frac{\partial}{\partial y_\alpha}$ for any $1 \leq \alpha \leq n$). Notice D is just defined locally. Define a new operator \mathcal{L} as (ϕ is any test function):

$$\mathcal{L}\phi = \sum_{\alpha,\beta=1}^{n+1} g'^{\alpha\bar{\beta}} \frac{\partial^2 \phi}{\partial z_\alpha \partial \bar{z}_\beta}$$

where $(g'^{\alpha\bar{\beta}}) = (g'_{\alpha\bar{\beta}})^{-1} = \left(g_{\alpha\bar{\beta}} + \frac{\partial^2 \varphi}{\partial z_\alpha \partial \bar{z}_\beta}\right)^{-1}$. Differentiating both side of equation (5) by D , we get

$$\mathcal{L} D\varphi = D \ln f + \sum_{\alpha,\beta=1}^{n+1} g'^{\alpha\bar{\beta}} D g_{\alpha\bar{\beta}}.$$

Thus there exists a constant C which depends only on $(V \times \mathbf{R}, g)$ such that

$$\mathcal{L}D(\varphi - \varphi_0) \leq C(1 + \sum_{\alpha=1}^{n+1} g'^{\alpha\bar{\alpha}}) \quad (8)$$

We will now employ a barrier function of the form

$$\nu = (\varphi - \varphi_0) + s(h - \varphi_0) - N \cdot x^2 \quad (9)$$

near the boundary point, and s, N are positive constants to be determined. We may take δ small enough so that x is small in $\Omega_\delta = (V \times \mathbf{R}) \cap B_\delta(0)$. The main essence of the proof is:

Lemma 4 *For N sufficiently large and s, δ sufficiently small, we have*

$$\mathcal{L} \nu \leq -\frac{\epsilon}{4}(1 + \sum_{\alpha=1}^{n+1} g'^{\alpha\bar{\alpha}}) \text{ in } \Omega_\delta, \quad \nu \geq 0 \text{ on } \partial\Omega_\delta.$$

Proof Since $g_{\alpha\bar{\beta}} + \frac{\partial^2 \varphi_0}{\partial z_\alpha \partial \bar{z}_\beta} \geq \epsilon \delta_{\alpha\bar{\beta}}$, we have

$$\mathcal{L}(\varphi - \varphi_0) = \sum_{\alpha,\beta=1}^{n+1} g'^{\alpha\bar{\beta}} [(g_{\alpha\bar{\beta}} + \frac{\partial^2 \varphi}{\partial z_\alpha \partial \bar{z}_\beta}) - (g_{\alpha\bar{\beta}} + \frac{\partial^2 \varphi_0}{\partial z_\alpha \partial \bar{z}_\beta})] \leq n+1 - \epsilon \sum_{\alpha=1}^{n+1} g'^{\alpha\bar{\alpha}}$$

and

$$\mathcal{L}(h - \varphi_0) \leq C_1(1 + \sum_{\alpha=1}^{n+1} g'^{\alpha\bar{\alpha}})$$

for some constant C_1 . Furthermore, $\mathcal{L} x^2 = 2g'^{(n+1)\overline{n+1}}$. Thus

$$\begin{aligned} \mathcal{L}\nu &= \mathcal{L}(\varphi - \varphi_0) + s \cdot \mathcal{L}(h - \varphi_0) - 2 \cdot N \cdot g'^{(n+1)\overline{n+1}} \\ &\leq n+1 - \epsilon \sum_{\alpha=1}^{n+1} g'^{\alpha\bar{\alpha}} + sC_1 + sC_1 \sum_{\alpha=1}^{n+1} g'^{\alpha\bar{\alpha}} - 2Ng'^{(n+1)\overline{n+1}}. \end{aligned}$$

Suppose $0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_{n+1}$ are eigenvalues of $(g'_{\alpha\bar{\beta}})_{(n+1)(n+1)}$. Thus

$$\sum_{\alpha=1}^{n+1} g'^{\alpha\bar{\alpha}} = \sum_{\alpha=1}^{n+1} \lambda_\alpha^{-1}, \quad g'^{(n+1)\overline{n+1}} \geq \lambda_n^{-1}.$$

Thus,

$$\begin{aligned} \frac{\epsilon}{4} \sum_{\alpha=1}^{n+1} g'^{\alpha\bar{\alpha}} + Ng'^{(n+1)\overline{n+1}} &\geq \frac{\epsilon}{4} \sum_{\alpha=1}^n \lambda_\alpha^{-1} + (N + \frac{\epsilon}{4}) \lambda_{n+1}^{-1} \\ &\geq (n+1) \frac{\epsilon}{4} N^{\frac{1}{n+1}} (\lambda_1 \cdot \lambda_2 \cdots \lambda_{n+1})^{-\frac{1}{n+1}} = C_2 N^{\frac{1}{n+1}}. \end{aligned}$$

Choose N large enough so that

$$-C_2 N^{\frac{1}{n+1}} + (n+1) + sC_1 < -\frac{\epsilon}{4}.$$

Choose s small enough so that $s \cdot C_1 \leq \frac{\epsilon}{4}$. Then

$$\mathcal{L}\nu \leq -\frac{\epsilon}{4}(1 + \sum_{\alpha=1}^{n+1} g'^{\alpha\bar{\alpha}}).$$

From now on we fix N . Observe that $\Delta(h - \varphi_0) < -2\epsilon$, then there exists a constant C_0 which depends only on g such that $h - \varphi_0 > C_0 x$ near $\partial(V \times \mathbf{R})$. Choose δ small enough so that

$$s(h - \varphi_0) - Nx^2 \geq (sC_0 - N\delta)x \geq 0.$$

Then $\nu \geq 0$ in $\partial\Omega_\delta$. QED.

Proof of Lemma 3: Let $M = \max(|\nabla\varphi|_g + 1)$. Choose $A \gg B \gg C, C_1$. In additional, choose A, B as a big multiple of M . Notice that $|D\varphi| \leq 2M$ in Ω_δ . For δ fixed as in Lemma 4, we have $B\delta^2 - |D(\varphi - \varphi_0)| > 0$. Consider $w = A\nu + B|z|^2 + D(\varphi - \varphi_0)$. Then $w \geq 0$ in $\partial\Omega_\delta$ and $w(0) = 0$. Moreover,

$$\mathcal{L}w \leq (-\frac{\epsilon A}{4} + 2B + C)(1 + \sum_{\alpha=1}^{n+1} g'^{\alpha\bar{\alpha}}) < 0.$$

Maximal Principal implies that $w \geq 0$ in Ω_δ . Since $w(0) = 0$, then $\frac{\partial w}{\partial x} \geq 0$. In other words,

$$\frac{\partial}{\partial x} D\varphi(0) < C_3 \cdot M$$

for some uniform constant C_3 . Since D is any 1st order constant operator near $\partial(V \times \mathbf{R})$. Replace D with $-D$, we get

$$-\frac{\partial}{\partial x} D\varphi(0) < C_3 \cdot M$$

On the other hand, since $\partial\mathbf{R}$ is given by $x = 0$ in our special case, we then have the trivial estimate:

$$\frac{\partial}{\partial y} D(\varphi - \varphi_0)(0) = 0.$$

Therefore,

$$|\frac{\partial}{\partial z_{n+1}} D\varphi(0)| < C_3 \cdot M$$

Lemma 3 follows from here directly. QED.

3.2 Blowing up analysis

Lemma 5 *Any bounded weakly sub-harmonic function in two dimensional plane is a constant.*

This is a standard fact in geometry analysis, we will omit the proof here. Notice this lemma is false if dimension is no less than 3.

The essence of blowing up analysis is to use “micro-scope” to analyze what happen in a small neighborhood via rescaling. Hence it doesn't make any difference what the global structure of background metric is, or what the metric is. Under rescaling, everything become Euclidean anyway. We may as well view the manifold as a domain in Euclidean space. we will use variable x to denote position in $V \times \mathbf{R}$.

Proof of theorem 2: Suppose $\frac{1}{\epsilon_i} = \max_{V \times \mathbf{R}} |\nabla\varphi_i|_g \rightarrow \infty$. We want to draw a contradiction from this statement.

Suppose $|\nabla\varphi_i|_g(x_i) = \frac{1}{\epsilon_i}$. By theorem 1, we have $\max_{V \times \mathbf{R}} \Delta\varphi_i \leq \frac{1}{\epsilon_i^2}$. Choose a convergent subsequence of x_i such that $x_i \rightarrow \underline{x}$. Choose a tiny neighborhood $B_\delta(\underline{x})$ of \underline{x} so that $g_{\alpha\bar{\beta}}(\underline{x}) = \delta_{\alpha\bar{\beta}}$ and g is essentially an identical matrix in $B_\delta(\underline{x})$. For simplicity, let us pretend that g is an Euclidean metric in $B_\delta(\underline{x})$. There are two cases to consider: the first case is when $\underline{x} \in \partial(V \times \mathbf{R})$ and the 2nd case is when \underline{x} is in the interior of $V \times \mathbf{R}$.

We define the blowing up sequence as

$$\tilde{\varphi}_i(x) = \varphi_i(x_i + \epsilon_i x), \forall x \in B_{\frac{\delta}{\epsilon_i}}(0).$$

Then $|\nabla\tilde{\varphi}_i(0)| = 1$ and

$$\max_{B_{\frac{\delta}{\epsilon_i}}(0)} |\nabla\tilde{\varphi}_i| \leq 1, \quad \text{and} \quad \max_{B_{\frac{\delta}{\epsilon_i}}(0)} |\Delta\tilde{\varphi}_i| \leq C.$$

Observe $\varphi_0 \leq \varphi_i \leq h$ ($\forall i$). Re-scale φ_0 and h accordingly:

$$\tilde{\varphi}_0(x) = \varphi_0(x_i + \epsilon_i x), \quad \tilde{h}(x) = h(x_i + \epsilon_i x), \quad \forall x \in B_{\frac{\delta}{\epsilon_i}}(0).$$

Thus $\lim_{i \rightarrow \infty} \tilde{\varphi}_0(x) = \varphi_0(\underline{x})$ and $\lim_{i \rightarrow \infty} \tilde{h}(x) = h(\underline{x})$. Moreover,

$$\tilde{\varphi}_0 \leq \tilde{\varphi}_i \leq \tilde{h}, \quad \forall i = 1, 2, \dots \quad (10)$$

There exists a subsequence of $\tilde{\varphi}_i$ and a limit function $\tilde{\varphi}$ in C^{n+1} (or half plane in case \underline{x} in the boundary) such that in any fixed ball of $B_l(0)$ (or half ball if \underline{x} is in the boundary) we have $\tilde{\varphi}_i \rightarrow \tilde{\varphi}$ in $C^{1,\eta}$ in the ball $B_l(0)$ (or half ball) for any $0 < \eta < 1$. This implies

$$|\nabla\tilde{\varphi}(0)| = 1. \quad (11)$$

In additional, inequality (10) holds in the limit:

$$\varphi_0(\underline{x}) \leq \tilde{\varphi}(x) \leq h(\underline{x}), \quad \forall x. \quad (12)$$

Case 1: Suppose $\underline{x} \in \partial(V \times \mathbf{R})$. Then $h(\underline{x}) = \varphi_0(\underline{x})$. Inequality (12) implies that $\tilde{\varphi}$ is a constant function in its domain. In particular, we have $|\nabla\tilde{\varphi}(x)| \equiv 0$. This contradicts our assertion (11). Thus the theorem is proved in this case.

Case 2: Suppose \underline{x} is in the interior of $V \times \mathbf{R}$. Then $\tilde{\varphi}(x)$ is a well defined $C^{1,\eta}$ and bounded function in C^{n+1} . We claim that this function is weakly sub-harmonic in any complex line through origin. If this claim is true, then Lemma 5 says it must be constant for any complex line through origin. Therefore, the function itself must be a constant as well. Thus $|\nabla\tilde{\varphi}| \equiv 0$. It again contradicts with our assertion (11). Thus the theorem is proved also, provided we can prove this claim.

Without loss of generality, we consider the complex line T is

$$z_2 = z_3 = \dots = z_{n+1} = 0.$$

Observed that (near \underline{x}) the following holds

$$0 < (\delta_{\alpha\bar{\beta}} + \frac{\partial^2 \varphi_i}{\partial z_\alpha \partial \bar{z}_\beta})_{(n+1)(n+1)} < \frac{C}{\epsilon_i^2} (\delta_{\alpha\bar{\beta}})_{(n+1)(n+1)}, \forall i.$$

After rescaling, we have

$$0 < \epsilon_i^2 \cdot (\delta_{\alpha\bar{\beta}})_{(n+1)(n+1)} + (\frac{\partial^2 \tilde{\varphi}_i}{\partial z_\alpha \partial \bar{z}_\beta})_{(n+1)(n+1)} < C \cdot (\delta_{\alpha\bar{\beta}})_{(n+1)(n+1)}.$$

Restricting this to a complex line T , we have

$$0 < \epsilon_i^2 + \frac{\partial^2 \tilde{\varphi}_i}{\partial z_1 \partial \bar{z}_1} < C \quad (13)$$

Thus one can choose a subsequence of $\tilde{\varphi}_i$ which converges $C^{1,\eta}$ ($0 < \eta < 1$) locally in T to some function ψ . Since the convergence is in $C^{1,\eta}$, thus $\psi = \tilde{\varphi}|_T$; i.e., ψ is the restriction of $\tilde{\varphi}$ in this complex line T . By taking weak limit in inequality (11), then $\tilde{\varphi}_i|_T$ weakly converge to ψ in $H_{loc}^{2,p}$ topology for any $p > 1$. Therefore, ψ is a weakly sub-harmonic function by taking weak limit in inequality (11). Therefore $\psi = \tilde{\varphi}|_T$ is a constant by Lemma 5. Our claim is then proved. QED.

4 Uniqueness of weak C^0 geodesic

Notation follows from previous section.

Definition 1 A function φ is generalized-pluri-subharmonic in $V \times \mathbf{R}$ if $\sum_{\alpha,\beta=1}^{n+1} (g_{\alpha\bar{\beta}} + \frac{\partial^2 \varphi}{\partial z_\alpha \partial \bar{z}_\beta}) dz_\alpha d\bar{z}_\beta$ defines a strictly positive Kähler metric in $V \times \mathbf{R}$.

Definition 2 A continuous function φ in $V \times \mathbf{R}$ is a weak C^0 solution to degenerated Monge-Ampere equation (3) with prescribing boundary data φ_0 if the following statement is true: $\forall \epsilon > 0$, there exists a pluri subharmonic function $\tilde{\varphi}$ in $V \times \mathbf{R}$ such that $|\varphi - \tilde{\varphi}| < \epsilon$ and $\tilde{\varphi}$ solves equation (5) with some positive function $0 < f < \epsilon$ at $t = 1$, and with the same boundary data φ_0 .

Clearly, the solution we obtain through continuous method is a weak C^0 solution of equation (3).

Theorem 4 Suppose φ_1, φ_2 are two C^0 weak solutions to the degenerated Monge-Ampere equation with prescribing boundary condition h_1, h_2 . Then

$$\max_{V \times \mathbf{R}} |\varphi_1 - \varphi_2| \leq \max_{\partial(V \times \mathbf{R})} |h_1 - h_2|.$$

Corollary 2 The solution to degenerated Monge-Ampere equation is unique as soon as the boundary data is fixed.

Proof: Suppose ϕ_1, ϕ_2 are two approximate generalized-pluri-subharmonic solutions of φ_1, φ_2 in the sense of definition 2. In other words

$$\det(g + \frac{\partial^2 \phi_i}{\partial z_\alpha \partial \bar{z}_\beta}) = f_i \cdot \det(g) > 0 \text{ in } V \times \mathbf{R}; \quad \text{and } \phi_i = h_i \text{ in } \partial(V \times \mathbf{R}), \quad i = 1, 2$$

such that $\max_{V \times \mathbf{R}} (|\varphi_1 - \phi_1| + f_1)$ and $\max_{V \times \mathbf{R}} (|\varphi_2 - \phi_2| + f_2)$ could be made as small as we wanted.

$\forall \epsilon > 0$, we want to show

$$\max_{V \times \mathbf{R}} (\varphi_1 - \varphi_2) \leq \max_{V \times \mathbf{R}} (h_1 - h_2) + 2\epsilon.$$

Choose f_1 such that $0 < f_1 < \epsilon$ and $\max_{V \times \mathbf{R}} |\varphi_1 - \phi_1| < \epsilon$. Choose f_2 such that $0 < f_2 \leq \frac{1}{2} \min_{V \times \mathbf{R}} f_1 < \epsilon$ and $\max_{V \times \mathbf{R}} |\varphi_2 - \phi_2| < \epsilon$. Then ϕ_1 is a sub-solution to ϕ_2 (thus $\phi_1 < \phi_2$) if $h_1 = h_2$. In general, we have

$$\max_{V \times \mathbf{R}} (\phi_1 - \phi_2) \leq \max_{\partial(V \times \mathbf{R})} (h_1 - h_2).$$

Thus

$$\begin{aligned} \max_{V \times \mathbf{R}} (\varphi_1 - \varphi_2) &= \max_{V \times \mathbf{R}} (\varphi_1 - \phi_1) + \max_{V \times \mathbf{R}} (\phi_1 - \phi_2) + \max_{V \times \mathbf{R}} (\phi_2 - \varphi_2) \\ &\leq \epsilon + \max_{\partial(V \times \mathbf{R})} (h_1 - h_2) + \epsilon \\ &= \max_{\partial(V \times \mathbf{R})} (h_1 - h_2) + 2\epsilon. \end{aligned}$$

Change the role of φ_1 and φ_2 , we obtain

$$\max_{V \times \mathbf{R}} (\varphi_2 - \varphi_1) \leq \max_{\partial(V \times \mathbf{R})} (h_2 - h_1) + 2\epsilon.$$

Thus

$$\max_{V \times \mathbf{R}} |\varphi_1 - \varphi_2| \leq \max_{\partial(V \times \mathbf{R})} |h_1 - h_2| + 2\epsilon.$$

Let $\epsilon \rightarrow 0$, we obtain the desired result. QED.

5 The space of Kähler metric is a metric space—Triangular inequality

In this section, we want to prove that the space of Kähler metric is a metric space and the $C^{1,1}$ geodesic between any two points realizes the global minimal length over all possible paths. To prove this claim, one inevitably need to take derivatives of lengths for a family of $C^{1,1}$ geodesics. However, the length for a $C^{1,1}$ geodesic is just barely defined (the integrand is in L^p space). In general, one can not take derivatives. Therefore, we must find ways to circumvent this trouble.

Definition 3 A path $\varphi(t)$ ($0 < t < 1$) in the space of Kähler metrics is a convex path if $\varphi(t)$ is a generalized-pluri-subharmonic function in $V \times (I \times S^1)$ (see definition 1).

Suppose $\text{vol}(t)$ ($0 \leq t \leq 1$) is a family of strictly positive volume form in V such that

$$\int_V \text{vol}(t) = \int_V \det g.$$

The notion of ϵ -approximate geodesic is defined with respect to such a volume form:

Definition 4 A convex path $\varphi(t)$ in the space of Kähler metrics is called ϵ -approximate geodesic if the following holds:

$$(\varphi'' - |\nabla \varphi'|_{g(t)}^2) \det g(t) = \epsilon \cdot \text{vol}(t)$$

where $g(t)_{\alpha\bar{\beta}} = g_{\alpha\bar{\beta}} + \frac{\partial^2 \varphi}{\partial z_\alpha \partial \bar{z}_\beta}$ ($1 \leq \alpha, \beta \leq n$).

Remark 2 The definition is really independent of these volume forms since we only care what happens when ϵ is really small. For convenience, sometimes we choose $\text{vol}(t) \equiv \det g$ (a volume form independent of t).

Lemma 6 Suppose $\varphi(t)$ ($0 \leq t \leq 1$) is an ϵ -approximate geodesics. Define the energy element as $E(t) = \int_V \varphi'(t)^2 d g(t)$. Then

$$\max_t \left| \frac{dE}{dt} \right| \leq 2\epsilon \cdot \max_{V \times I} |\varphi'(t)| \cdot M$$

where $M = \int_V \det g$ is the total volume of V which depends only on the Kähler class.

Proof:

$$\begin{aligned}
\left| \frac{dE}{dt} \right| &= \left| \int_V (2\varphi''\varphi' + \varphi'^2 \Delta_{g(t)}\varphi') dg(t) \right| \\
&= 2 \left| \int_V \varphi'(\varphi'' - \frac{1}{2}|\nabla\varphi'|_{g(t)}^2) \det g(t) \right| \quad \text{QED.} \\
&= 2 \left| \int_V \varphi' \epsilon \operatorname{vol}(t) \right| \leq 2\epsilon \cdot \max_{V \times I} |\varphi'(t)| \cdot M.
\end{aligned}$$

Proposition 2 Suppose $\varphi(t)$ is a $C^{1,1}$ geodesic in \mathcal{H} from 0 to φ and $I(\varphi) = 0$. Then the following inequality holds

$$\int_0^1 \sqrt{\int_V \varphi'^2 d\mu_{\varphi_t}} dt \geq M^{-1} \left(\max \left(\int_{\varphi>0} \varphi d\mu_{\varphi}, -\int_{\varphi<0} \varphi d\mu_0 \right) \right).$$

In other words, the length of any $C^{1,1}$ geodesic is strictly positive.

Proof: As in definition (4), suppose $\varphi(\epsilon, t)$ is a ϵ -approximated geodesic between 0 and φ . (We will drop the dependence of ϵ in this proof since no confusion shall arise from this omission). First of all, from definition of ϵ -approximated geodesic, we have

$$\varphi'' - \frac{1}{2}|\nabla\varphi'|_{g(t)}^2 > 0.$$

In particular, we have $\varphi''(t) \geq 0$. Thus

$$\varphi'(0) \leq \varphi \leq \varphi'(1). \quad (14)$$

Consider $f(t) = I(t\varphi)$, $t \in [0, 1]$. Then $f'(t) = \int_V \varphi d\mu_{t\varphi}$ and

$$f''(t) = \int_V \varphi \Delta_{g(t\varphi)} \varphi d\mu_{t\varphi} \leq 0.$$

Thus, we have $f'(0) \geq \frac{f(1)-f(0)}{1-0} \geq f'(1)$. In other words, we have

$$\int_V \varphi d\mu_0 \geq I(\varphi) \geq \int_V \varphi d\mu_{\varphi}.$$

Since we assume $I(\varphi) = 0$, and φ not identically zero, then it must take both positive and negative values. Then the length (or energy) of the geodesic is given by

$$E = \int_V \varphi'^2 d\mu_{\varphi_t},$$

for any $t \in [0, 1]$. In particular, taking $t = 1$,

$$\sqrt{E(1)} \geq M^{-1/2} \int_V |\varphi'(1)| d\mu_{\varphi} > M^{-1/2} \int_{\varphi'(1)>0} \varphi'(1) d\mu_{\varphi},$$

where M is the volume of V (which is of course the same for all metrics in \mathcal{H}). It follows from inequality (14) that

$$\int_{\varphi'(1)>0} \varphi' d\mu_{\varphi} \geq \int_{\varphi>0} \varphi d\mu_{\varphi},$$

where the last term is strictly positive by the remarks above, and depends only on φ and not on the geodesic. A similar argument gives

$$\sqrt{E(0)} > -M^{-1/2} \int_{\varphi<0} \varphi d\mu_0.$$

The previous lemma implies that for any $t_1, t_2 \in [0, 1]$, we have

$$|E(t_1) - E(t_2)| < C \cdot \epsilon$$

for some constant C independent of ϵ . Thus

$$\sqrt{E(t)} \geq M^{-1/2} \max\left(\int_{\varphi>0} \varphi d\mu_\varphi, -\int_{\varphi<0} \varphi d\mu_0\right) - C \cdot \epsilon.$$

Now integrating from $t = 0$ to 1 and let $\epsilon \rightarrow 0$. Then

$$\int_0^1 \sqrt{\int_V \varphi'^2 d\mu_\varphi} dt \geq M^{-1/2} \max\left(\int_{\varphi>0} \varphi d\mu_\varphi, -\int_{\varphi<0} \varphi d\mu_0\right).$$

Then this proposition is proved. *QED*.

Remark 3 *This proposition verifies Donaldson's 2nd conjecture **. However, it will not imply \mathcal{H} is a metric space automatically since the geodesic is not sufficiently differentiable. However, one can easily verify that $C^{1,1}$ geodesic minimizes length over all possible convex curves between the two end points. To show that it minimizes length over all possible curves, not just convex ones, we need to prove that the triangular inequality is satisfied by the geodesic distance (see definition below).*

Definition 5 *Let φ_1, φ_2 be two distinct points in the space of metrics. According to theorem 3 and Corollary 2, there exists a unique geodesic connecting these two points. Define the geodesic distance as the length of this geodesic. Denoted as $d(\varphi_1, \varphi_2)$.*

Theorem 5 *Suppose $C : \varphi(s) : [0, 1] \rightarrow \mathcal{H}$ is a smooth curve in \mathcal{H} . Suppose p is a base point of \mathcal{H} . For any s , the geodesic distance from p to $\varphi(s)$ is no greater than the sum of geodesic distance from p to $\varphi(0)$ and the length from $\varphi(0)$ to $\varphi(s)$ along this curve C . In particular, if $C : \varphi(s) : [0, 1] \rightarrow \mathcal{H}$ is a geodesic, then the geodesic distance satisfies:*

$$d(0, \varphi(1)) \leq d(0, \varphi(0)) + d(\varphi(0), \varphi(1)).$$

Lemma 7 *(Geodesic approximation lemma): Suppose $C_i : \varphi_i(s) : [0, 1] \rightarrow \mathcal{H} (i = 1, 2)$ are two smooth curves in \mathcal{H} . For ϵ_0 small enough, there exist two parameters smooth families of curves $C(s, \epsilon) : \phi(t, s, \epsilon) : [0, 1] \times [0, 1] \times (0, \epsilon_0] (0 \leq t, s \leq 1, 0 < \epsilon \leq \epsilon_0)$ such that the following properties hold:*

1. *For any fixed s and ϵ , $C(s, \epsilon)$ is a ϵ -approximate geodesic from $\varphi_1(s)$ to $\varphi_2(s)$. More precisely, $\phi(z, t, s, \epsilon)$ solves the corresponding Monge-Ampere equation:*

$$\det\left(g + \frac{\partial^2 \phi}{\partial z_\alpha \partial \bar{z}_\beta}\right) = \epsilon \cdot \det(g) \text{ in } V \times \mathbf{R}; \quad \text{and } \phi(z', 0, s, \epsilon) = \varphi_1(z', s), \phi(z', 1, s, \epsilon) = \varphi_2(z', s). \quad (15)$$

Here we follows notation in section 3, and $z_{n+1} = t + \sqrt{-1}\theta$ where depends of ϕ on θ is trivial.

2. *There exists a uniform constant C (which depends only on φ_1, φ_2 such that*

$$|\phi| + \left|\frac{\partial \phi}{\partial s}\right| + \left|\frac{\partial \phi}{\partial t}\right| < C; \quad 0 \leq \frac{\partial^2 \phi}{\partial t^2} < C, \quad \frac{\partial^2 \phi}{\partial s^2} < C.$$

3. *For fixed s , let $\epsilon \rightarrow 0$, the convex curve $C(s, \epsilon)$ converges to the unique geodesic between $\varphi_1(s)$ and $\varphi_2(s)$ in weak $C^{1,1}$ topology.*

** In [14], Donaldson provided a formal proof to this proposition after assuming the existence of a smooth geodesic between any two metrics. Our proof follows his idea closely.

4. Define energy element along $C(s, \epsilon)$ by

$$E(t, s, \epsilon) = \int_V \left| \frac{\partial \phi}{\partial t} \right|^2 d g(t, s, \epsilon)$$

where $g(t, s, \epsilon)$ is the corresponding Kähler metric define by $\phi(t, s, \epsilon)$. Then there exists a uniform constant C such that

$$\max_{t,s} \left| \frac{\partial E}{\partial t} \right| \leq \epsilon \cdot C \cdot M.$$

In other words, the energy/length element converges to a constant along each convex curve if $\epsilon \rightarrow 0$.

Proof : Everything follows from theorem 3 and 4 and lemma 6 except the bound on $|\frac{\partial \phi}{\partial s}|$ and a upbound on $\frac{\partial^2 \phi}{\partial s^2}$ which follow from maximal principal directly since

$$\mathcal{L}\left(\frac{\partial \phi}{\partial s}\right) = 0$$

and

$$\mathcal{L}\left(\frac{\partial^2 \phi}{\partial s^2}\right) = \text{tr}_{g'} \{ \text{Hess} \frac{\partial \phi}{\partial s} \cdot \text{Hess} \frac{\partial \phi}{\partial s} \} \geq 0. \quad \text{QED.}$$

Proof of theorem 5: Apply geodesic approximation lemma with special case that $\varphi_1(s) \equiv p$. We follow notations in the previous lemma. For ϵ_0 small enough, there exist two parameters smooth families of curves $C(s, \epsilon) : \phi(t, s, \epsilon) : [0, 1] \times [0, 1] \times (0, \epsilon_0] (0 \leq t, s \leq 1, 0 < \epsilon \leq \epsilon_0)$ such that

$$\det(g + \frac{\partial^2 \phi}{\partial z_\alpha \partial \bar{z}_\beta}) = \epsilon \cdot \det(g), \text{ in } V \times \mathbf{R}; \quad \text{and } \phi(z', 0, s, \epsilon) = 0, \phi(z', 1, s, \epsilon) = \varphi(z', s).$$

Denote the length of the curve $\phi(t, s, \epsilon)$ from p to $\varphi(s)$ as $L(s, \epsilon)$, denote the geodesic distance between p and $\varphi(s)$ as $L(s)$, and denote the length from $\varphi(0)$ to $\varphi(s)$ along curve C as $l(s)$.

Clearly, $l(s) = \int_0^s \sqrt{\int_V \left| \frac{\partial \varphi}{\partial \tau} \right|^2 d g(\tau)} d \tau$ where $g(\tau)$ is the Kähler metric defined by $\varphi(\tau)$, and

$$L(s, \epsilon) = \int_0^1 \sqrt{E(t, s, \epsilon)} dt = \int_0^1 \sqrt{\int_V \left| \frac{\partial \phi}{\partial t} \right|^2 d g(t, s, \epsilon)} dt, \text{ and } \lim_{\epsilon \rightarrow 0} L(s, \epsilon) = L(s).$$

Define $F(s, \epsilon) = L(s, \epsilon) + l(s)$ and $F(s) = L(s) + l(s)$. What we need to prove is : $F(1) \geq F(0)$. This will be done if we can show that $F'(s) \geq 0, \forall s \in [0, 1]$. The last statement would be straightforward if the deformation of geodesics is C^1 . Since we don't have it, we need to take derivatives on $F(s, \epsilon)$ for $\epsilon > 0$ instead. Notice $\frac{\partial \phi}{\partial s} = 0$ at $t = 0$ in the following deduction:

$$\begin{aligned} \frac{dL(s, \epsilon)}{ds} &= \int_0^1 \frac{1}{2} E(t, s, \epsilon)^{-\frac{1}{2}} \int_V \left(2 \frac{\partial \phi}{\partial t} \frac{\partial^2 \phi}{\partial t \partial s} + \left(\frac{\partial \phi}{\partial t} \right)^2 \Delta_{g(t, s, \epsilon)} \frac{\partial \phi}{\partial s} \right) d g(t, s, \epsilon) dt \\ &= \int_0^1 E(t, s, \epsilon)^{-\frac{1}{2}} \left\{ \frac{\partial}{\partial t} \left(\int_V \frac{\partial \phi}{\partial t} \frac{\partial \phi}{\partial s} d g(t, s, \epsilon) \right) - \int_V \frac{\partial \phi}{\partial s} \left(\frac{\partial^2 \phi}{\partial t^2} - \frac{1}{2} |\nabla \frac{\partial \phi}{\partial t}|^2 \right) d g(t, s, \epsilon) \right\} dt \\ &= \{ E(t, s, \epsilon)^{-\frac{1}{2}} \int_V \frac{\partial \phi}{\partial t} \frac{\partial \phi}{\partial s} d g(t, s, \epsilon) \} \Big|_0^1 - \int_0^1 \{ E(t, s, \epsilon)^{-\frac{1}{2}} \int_V \frac{\partial \phi}{\partial s} \left(\frac{\partial^2 \phi}{\partial t^2} - \frac{1}{2} |\nabla \frac{\partial \phi}{\partial t}|^2 \right) d g(t, s, \epsilon) \} dt \\ &\quad + \int_0^1 \{ E(t, s, \epsilon)^{-\frac{3}{2}} \int_V \frac{\partial \phi}{\partial t} \frac{\partial \phi}{\partial s} d g(t, s, \epsilon) \cdot \int_V \frac{\partial \phi}{\partial t} \left(\frac{\partial^2 \phi}{\partial t^2} - \frac{1}{2} |\nabla \frac{\partial \phi}{\partial t}|^2 \right) d g(t, s, \epsilon) \} dt \\ &= \int_V \frac{\partial \phi(1, s, \epsilon)}{\partial t} \frac{d \varphi}{ds} d g(s) \cdot \left\{ \int_V \left| \frac{\partial \phi(1, s, \epsilon)}{\partial t} \right|^2 d g(s) \right\}^{-\frac{1}{2}} - \int_0^1 \{ E(t, s, \epsilon)^{-\frac{1}{2}} \int_V \frac{\partial \phi}{\partial s} \epsilon \cdot \det g \} dt \\ &\quad + \int_0^1 \{ E(t, s, \epsilon)^{-\frac{3}{2}} \int_V \frac{\partial \phi}{\partial t} \frac{\partial \phi}{\partial s} d g(t, s, \epsilon) \cdot \int_V \frac{\partial \phi}{\partial t} \epsilon \cdot \det g \} dt. \end{aligned}$$

Observe that By Schwartz inequality, we have

$$\frac{dl(s)}{ds} = \sqrt{\int_V \left| \frac{\partial \varphi}{\partial s} \right|^2 dg(s)} \geq - \int_V \frac{\partial \phi(1, s, \epsilon)}{\partial t} \frac{d\varphi}{ds} dg(s) \cdot \left\{ \int_V \left| \frac{\partial \phi(1, s, \epsilon)}{\partial t} \right|^2 dg(s) \right\}^{-\frac{1}{2}}.$$

Observe that $F(s, \epsilon) = L(s, \epsilon) + l(s)$. Thus

$$\frac{dF(s, \epsilon)}{ds} \geq - \int_0^1 \{E(t, s, \epsilon)^{-\frac{1}{2}} \int_V \frac{\partial \phi}{\partial s} \epsilon \cdot \det g\} dt + \int_0^1 \{E(t, s, \epsilon)^{-\frac{3}{2}} \int_V \frac{\partial \phi}{\partial t} \frac{\partial \phi}{\partial s} dg(t, s, \epsilon)\} \cdot \int_V \frac{\partial \phi}{\partial t} \epsilon \cdot \det g\} dt.$$

Integrating from 0 to $s \in (0, 1]$, we obtain

$$\begin{aligned} F(s, \epsilon) - F(0, \epsilon) &\geq - \int_0^s \int_0^1 \{E(t, \tau, \epsilon)^{-\frac{1}{2}} \int_V \frac{\partial \phi}{\partial \tau} \epsilon \cdot \det g\} dt d\tau \\ &\quad + \int_0^s \int_0^1 \{E(t, \tau, \epsilon)^{-\frac{3}{2}} \int_V \frac{\partial \phi}{\partial t} \frac{\partial \phi}{\partial \tau} dg(t, \tau, \epsilon)\} \cdot \int_V \frac{\partial \phi}{\partial t} \epsilon \cdot \det g\} dt d\tau \\ &\geq -C\epsilon \end{aligned}$$

for some big constant C depends only on $(V \times \mathbf{R}, g)$ and the initial curve $C : \varphi(s) : [0, 1] \rightarrow \mathcal{H}$. Now take limit as $\epsilon \rightarrow 0$, we have $F(s) \geq F(0)$. In other words, the geodesic distance from p to $\varphi(s)$ is no greater than the sum of geodesic distance from p to $\varphi(0)$ and the length from $\varphi(0)$ to $\varphi(s)$ along this curve C . QED.

Corollary 3 *The geodesic distance between any two metrics realize the absolute minimum of the lengths over all possible paths.*

Proof: For any smooth curve $C : \varphi(s) : [0, 1] \rightarrow \mathcal{H}$, we want to show that the geodesic distance between the two end points $\varphi(0)$ and $\varphi(1)$ is no greater than the length of C . However, this follows directly from Theorem 5 by taking $p = \varphi(1)$ and $s = 1$. QED.

Theorem 6 *For any two Kähler potentials φ_1, φ_2 , the minimal length $d(\varphi_1, \varphi_2)$ over all possible paths which connect these two Kähler potentials is strictly positive, as long as $\varphi_1 \neq \varphi_2$. In other words, (\mathcal{H}, d) is a metric space. Moreover, the distance function is at least C^1 .*

Proof Immediately from Corollary 3 and proposition 2, we imply that (\mathcal{H}, d) is a metric space. Now we want to prove the differentiability of distance function. From the Proof of theorem 5, we have

$$\begin{aligned} \frac{dL(s, \epsilon)}{ds} &= \int_V \frac{\partial \phi(1, s, \epsilon)}{\partial t} \frac{d\varphi}{ds} dg(s) \cdot \left\{ \int_V \left| \frac{\partial \phi(1, s, \epsilon)}{\partial t} \right|^2 dg(s) \right\}^{-\frac{1}{2}} - \int_0^1 \{E(t, s, \epsilon)^{-\frac{1}{2}} \int_V \frac{\partial \phi}{\partial s} \epsilon \cdot \det g\} dt \\ &\quad + \int_0^1 \{E(t, s, \epsilon)^{-\frac{3}{2}} \int_V \frac{\partial \phi}{\partial t} \frac{\partial \phi}{\partial s} dg(t, s, \epsilon)\} \cdot \int_V \frac{\partial \phi}{\partial t} \epsilon \cdot \det g\} dt. \end{aligned}$$

Integrating this from s_1 to s_2 and divide by $s_2 - s_1$, we have

$$\begin{aligned} &\left| \frac{L(s_2, \epsilon) - L(s_1, \epsilon)}{s_2 - s_1} - \frac{1}{s_2 - s_1} \int_{s_1}^{s_2} \int_V \frac{\partial \phi(1, s, \epsilon)}{\partial t} \frac{d\varphi}{ds} dg(s) \cdot \left\{ \int_V \left| \frac{\partial \phi(1, s, \epsilon)}{\partial t} \right|^2 dg(s) \right\}^{-\frac{1}{2}} ds \right| \\ &\leq \frac{1}{s_2 - s_1} \int_{s_1}^{s_2} \int_0^1 \{E(t, s, \epsilon)^{-\frac{1}{2}} \int_V \left| \frac{\partial \phi}{\partial s} \right| \epsilon \cdot \det g\} dt ds \\ &\quad + \frac{1}{s_2 - s_1} \int_{s_1}^{s_2} \int_0^1 \{E(t, s, \epsilon)^{-\frac{3}{2}} \int_V \left| \frac{\partial \phi}{\partial t} \right| \left| \frac{\partial \phi}{\partial s} \right| dg(t, s, \epsilon)\} \cdot \int_V \left| \frac{\partial \phi}{\partial t} \right| \epsilon \cdot \det g\} dt ds \\ &\leq C\epsilon. \end{aligned}$$

Let $\epsilon \rightarrow 0$, and then let $s_2 \rightarrow s_1$ we have

$$\begin{aligned} \lim_{s_2 \rightarrow s_1} \frac{L(s_2) - L(s_1)}{s_2 - s_1} &= \lim_{s_2 \rightarrow s_1} \frac{1}{s_2 - s_1} \int_{s_1}^{s_2} \int_V \frac{\partial \phi(1, s)}{\partial t} \frac{d\varphi}{ds} dg(s) \cdot \left\{ \int_V \left| \frac{\partial \phi(1, s)}{\partial t} \right|^2 dg(s) \right\}^{-\frac{1}{2}} ds \\ &= \int_V \frac{\partial \phi(1, s)}{\partial t} \frac{d\varphi}{ds} dg(s) \cdot \left\{ \int_V \left| \frac{\partial \phi(1, s)}{\partial t} \right|^2 dg(s) \right\}^{-\frac{1}{2}}. \end{aligned}$$

The distance function L is then a differentiable function. QED.

6 Application: Uniqueness of Extremal Kähler metrics if $C_1(V) < 0$ and $C_1(V) = 0$

In this section, we want to show that if $C_1(V) < 0$, or if $C_1(V) = 0$, then extremal Kähler metric is unique in any Kähler class. Furthermore, if $C_1(V) \leq 0$, extremal Kähler metric (if existed) realizes the global minimum of Mabuchi energy functional in any Kähler class, thus gave a affirmative answer to a question raised by Tian Gang in this special case.

6.1 Uniqueness of c.s.c metric when $C_1(V) = 0$ and the lower bound of Mabuchi energy for $C_1(V) \leq 0$

We should now introduce an important operator—Lichernowicz operator \mathcal{D} . For any function h , $\mathcal{D}h = h_{,\alpha\beta}dz^\alpha \otimes dz^\beta$. If $\mathcal{D}h = 0$, then $\uparrow \bar{\partial}h = g^{\alpha\bar{\beta}} \frac{\partial h}{\partial \bar{\beta}} \frac{\partial}{\partial z^\alpha}$ is a holomorphic vector field. Now let us introduce Mabuchi functional. Like I_ρ, I , it is again defined by its derivatives and one should check it is well defined by verifying the second derivatives is symmetric (we will leave this to the reader). Let R be the scalar curvature of metric $g = g_0 + \sqrt{-1}\partial\bar{\partial}\varphi$ and \underline{R} be the average scalar curvature in the cohomology class, let $\psi \in T_\varphi\mathcal{H}$. Then the variation of Mabuchi energy of g at direction ψ is:

$$\delta_\psi E = - \int_V (R - \underline{R}) \cdot \psi \det g.$$

Along any smooth geodesic $\varphi(t) \in \mathcal{H}$, S. Donaldson shows

$$\frac{d^2 E}{dt^2} = \int_V |\mathcal{D}\varphi'(t)|_g^2 \det g.$$

Using this, Donaldson shows that constant curvature metric is unique in each Kähler class if the smooth geodesic conjecture is true. Now we want to prove the uniqueness of constant curvature metric in each Kähler class when $C_1(V) < 0$ or $C_1(V) = 0$, despite the fact we have not proven the smooth geodesic conjecture yet.

Theorem 7 *If either $C_1(V) < 0$ or $C_1(V) = 0$, then the constant curvature metric (if existed) in any Kähler class must be unique.*

Proof: Notations follow from section 5. Suppose $\varphi(t)$ is a ϵ - approximate geodesic. Then

$$\det g (\varphi'' - \frac{1}{2}|\nabla\varphi'|_g^2) = \epsilon \cdot \det h$$

where h is a given metrics in the Kähler class such that $Ric(h) < -c h$ if $C_1(V) < 0$ and $Ric(h) \equiv 0$ if $C_1(V) = 0$. Let $f = \varphi'' - \frac{1}{2}|\nabla\varphi'|_g^2 \geq 0$. Then

$$\nabla \ln \frac{\det g}{\det h} = -\nabla \ln f \quad (16)$$

and

$$\frac{d}{dt} \left(\int_V \varphi'(t) \det g \right) = \int_V f \cdot \det g = \epsilon \cdot \int_V \det h. \quad (17)$$

Let E denote the Mabuchi energy functional. Then

$$\frac{dE}{dt} = - \int_V (R - \underline{R}) \cdot \psi \det g$$

A direct calculation yields

$$\frac{d^2 E}{dt^2} = \int_V |\mathcal{D}\varphi'(t)|_g^2 \det g - \int_V (\varphi'' - \frac{1}{2}|\nabla\varphi'|_g^2) \cdot R \det g + \epsilon \cdot \underline{R} \cdot \int_V \det h. \quad (18)$$

where we already use equation (17). Now the 2nd term in the right hand side of above equation is:

$$\begin{aligned}
-\int_V R \cdot f \det g &= \int_V \Delta_g \ln \det g \cdot f \cdot \det g \\
&= \int_V \Delta_g \frac{\ln \det g}{\ln \det h} \cdot f \cdot \det g + \int_V \Delta_g \ln \det h \cdot f \cdot \det g \\
&= -\int_V \nabla_g \frac{\ln \det g}{\ln \det h} \cdot \nabla \ln f \det g - \int_V \text{tr}_g(\text{Ric}(h)) f \det g \\
&= \int_V |\nabla f|_g^2 \frac{1}{f} \det g - \int_V \text{tr}_g(\text{Ric}(h)) f \det g.
\end{aligned}$$

Thus integrate from $t = 0$ to 1 ,

$$\int_{V \times I} |\mathcal{D}\varphi'|_g^2 \det g \, dt + \int_{V \times I} \frac{|\nabla f|^2}{f} \det g \, dt - \int_{V \times I} \text{tr}_g(\text{Ric}(h)) f \det g \, dt = \frac{dE}{dt} \Big|_0^1 - \epsilon \underline{R} \cdot \int_V \det h \, dt. \quad (19)$$

If $\varphi(0)$ and $\varphi(1)$ are both of constant scalar curvature metrics, then $\frac{dE}{dt} \Big|_0^1 = 0$ and

$$\int_{V \times I} |\mathcal{D}\varphi'|_g^2 \det g \, dt + \int_{V \times I} \frac{|\nabla f|^2}{f} \det g \, dt - \int_{V \times I} \text{tr}_g(\text{Ric}(h)) f \det g \, dt = -\epsilon \underline{R} \cdot \int_V \det h \, dt. \quad (20)$$

Observe that $f \det g = \epsilon \cdot \det h$. We then imply from previous equation

$$\int_{V \times I} \frac{|\mathcal{D}\varphi'|_g^2}{f} \det h + \int_{V \times I} |\nabla \ln f|_g^2 \det h - \int_{V \times I} \text{tr}_g(\text{Ric}(h)) \det h = -\underline{R} \int_V \det h.$$

Clearly, if $C_1(V) = 0$, then $\underline{R} = 0$. Thus

$$\int_{V \times I} \frac{|\mathcal{D}\varphi'|_g^2}{f} \det h \, dt + \int_{V \times I} |\nabla \ln f|_g^2 \det h \, dt = 0$$

This easily implies that $\mathcal{D}\varphi(t) \equiv 0$ and $\uparrow \bar{\partial}\varphi'(t)$ is a holomorphic vector field. Since $C_1 = 0$, the only holomorphic vector field is constant vector field. Thus $\varphi'(t)$ is constant on V direction. In other words, $\varphi'(t)$ is a functional of t only. Hence, there exist at most one constant scalar curvature metric in each Kähler class when $C_1 = 0$. We postpone the proof of case $C_1 < 0$ to the next subsection.

Theorem 8 *If $C_1(V) \leq 0$, then constant scalar curvature metric, if existed, realizes the global minimum of Mabuchi energy functional in each Kähler class. In other words, if Mabuchi energy doesn't have a lower bound, then there exists no constant curvature metric in that cohomology class.*

Proof Suppose $\varphi_0 \in \mathcal{H}$ is a metric of constant curvature, then

$$\frac{dE}{dt} \Big|_{\varphi_0} = - \int_V (R - \underline{R}) \cdot \psi \det g = 0$$

For any metric $\varphi(1)$, let $\varphi(t) (0 \leq t \leq 1)$ is a path in \mathcal{H} which connects between $\varphi(0)$ and $\varphi(1)$. In additional, let us assume this is a ϵ - approximate geodesic where $\epsilon > 0$ may be chosen arbitrary small. From equation (17), we have

$$\begin{aligned}
\frac{d^2 E}{dt^2} &= \int_V |\mathcal{D}\varphi'(t)|_g^2 \det g - \int_V (\varphi'' - \frac{1}{2} |\nabla \varphi'|_g^2) \cdot R \det g + \epsilon \cdot \underline{R} \cdot \int_V \det h \\
&= \int_V |\mathcal{D}\varphi'(t)|_g^2 \det g + \int_V |\nabla f|_g^2 \frac{1}{f} \det g - \int_V \text{tr}_g(\text{Ric}(h)) f \det g + \epsilon \cdot \underline{R} \cdot \int_V \det h \\
&> -C\epsilon.
\end{aligned}$$

The last inequality holds since the average of scalar curvature is a topological invariant. Thus

$$E(t) - E(0) \geq -C\epsilon \frac{t^2}{2}, \quad \forall t \in [0, 1].$$

In particular, this holds for $t = 1$

$$E(\varphi(1)) - E(\varphi(0)) = E(1) - E(0) \geq -\frac{C \cdot \epsilon}{2}.$$

Now let $\epsilon \rightarrow 0$, we have

$$E(\varphi(1)) \geq E(\varphi(0)).$$

Thus the theorem is proved since $\varphi(1)$ is arbitrary chosen.

6.2 Uniqueness of c.s.c. metric when $C_1 < 0$

Now we turn our attentions to the case $C_1 < 0$. By initial assumption, $Ric(h) < -ch$ for some positive constant $c > 0$. Thus

$$\int_{V \times I} \frac{|\mathcal{D}\varphi'|_g^2}{f} \det h + \int_{V \times I} |\nabla \ln f|_g^2 \det h + c \cdot \int_{V \times I} \text{tr}_g(h) \det h \leq C (= -\underline{R} \int_V \det h). \quad (21)$$

We want to show that in the limit as $\epsilon \rightarrow 0$, we still have $\mathcal{D}\varphi'(t) = 0$. Let us first get an integral estimate on $f^{\frac{q}{2-q}} (1 < q < 2)$ with respect to measure $\det h \, dt$:

$$\begin{aligned} \int_{V \times I} f^{\frac{q}{2-q}} \det h \, dt &\leq C \cdot \int_{V \times I} f \det h \, dt \\ &\leq C \cdot \int_{V \times I} \left\{ f \cdot \frac{\det g}{\det h} \right\}^{\frac{1}{n}} \cdot \left\{ \frac{\det h}{\det g} \right\}^{\frac{1}{n}} \det h \, dt \\ &\leq \epsilon^{\frac{1}{n}} \int_{V \times I} \left\{ \frac{\det h}{\det g} \right\}^{\frac{1}{n}} \det h \, dt \\ &\leq C \cdot \epsilon^{\frac{1}{n}} \int_{V \times I} \text{tr}_g(h) \det h \, dt \rightarrow 0. \end{aligned}$$

Let $X = \uparrow \bar{\partial}\varphi'(t) = g^{\alpha\bar{\beta}} \frac{\partial\varphi'}{\partial z_{\bar{\beta}}} \frac{\partial}{\partial z_{\alpha}}$. Then we want to show that X is uniformly in L^2 with respect to measure $h + dt^2$.

$$\begin{aligned} \int_{V \times I} |X|_h^2 \det h \, dt &= \int_{V \times I} \sum_{\alpha, \beta} h_{\alpha\bar{\beta}} X^{\alpha} \overline{X^{\beta}} \det h \, dt \\ &= \int_{V \times I} \sum_{\alpha, \beta, \gamma, \delta} h_{\alpha\bar{\beta}} g^{\alpha\bar{\gamma}} \frac{\partial\varphi'}{\partial z_{\bar{\gamma}}} \overline{\left\{ g^{\beta\bar{\delta}} \frac{\partial\varphi'}{\partial z_{\bar{\delta}}} \right\}} \det h \, dt \\ &= \int_{V \times I} \sum_{\alpha, \beta, \gamma, \delta} h_{\alpha\bar{\beta}} g^{\alpha\bar{\gamma}} g^{\delta\bar{\beta}} \frac{\partial\varphi'}{\partial z_{\bar{\gamma}}} \frac{\partial\varphi'}{\partial z_{\delta}} \det h \, dt \\ &\leq \int_{V \times I} \text{tr}_g(h) |\nabla\varphi'|_g^2 \det h \\ &\leq C \cdot \int_{V \times I} \text{tr}_g(h) \det h \, dt \leq C. \end{aligned}$$

The second to last inequality holds since $f = \varphi'' - \frac{1}{2} |\nabla\varphi'|_g^2 \geq 0$ and $\varphi'' < C$. Thus $X \in L^2(V \times I)$ has a uniform upbound for the L^2 norm.

Consider $|\mathcal{D}\varphi'|_g$ as a function in $L^2(V \times I)$. First of all, it has a weak limit in $L^2(V \times I)$; secondly, its $L^q (1 < q < 2)$ norm tends to 0 as $\epsilon \rightarrow 0$.

$$\begin{aligned} \int_{V \times I} |\mathcal{D}\varphi'|_g^q \det h \, dt &= \int_{V \times I} \frac{|\mathcal{D}\varphi'|_g^q}{f^l} \cdot f^l \det h \, dt \\ &\leq \left(\int_{V \times I} \frac{|\mathcal{D}\varphi'|_g^{sq}}{f^{ls}} \det h \, dt \right)^{\frac{1}{s}} \cdot \left(\int_{V \times I} f^{l\tau} \det h \, dt \right)^{\frac{1}{\tau}} \quad \left(\text{where } \frac{1}{s} + \frac{1}{\tau} = 1 \right). \end{aligned}$$

Now l is some number we should choose appropriately:

$$ls = 1; qs = 2; \frac{1}{s} + \frac{1}{\tau} = 1.$$

Thus for any $q < 2$, we have

$$s = \frac{2}{q}; l = \frac{q}{2}; \tau = \frac{2}{2-q}.$$

Thus the above inequality reduce to

$$\int_{V \times I} |\mathcal{D}\varphi'|_g^q \det h \, dt \leq C \cdot \left(\int_{V \times I} \frac{|\mathcal{D}\varphi'|_g^2}{f} \det h \, dt \right)^{\frac{2}{q}} \cdot \left(\int_{V \times I} f^{\frac{q}{2-q}} \det h \, dt \right)^{\frac{(2-q)}{2}} \rightarrow 0.$$

For any vector $Y \in T^{1,0}(V \times I)$ (i.e., $Y = \sum_{i=1}^n Y^i \frac{\partial}{\partial z_i}$ where z_1, z_2, \dots, z_n are all of the coordinate functions in a local chart in V . We use $\frac{\partial Y}{\partial \bar{z}}$ to denote the vector valued (0,1) form $\sum_{i,j=1}^n \frac{\partial Y^i}{\partial \bar{z}_j} \frac{\partial}{\partial z_i} \otimes d\bar{z}_j$.) For a scalar function ψ in $V \times I$, denote $\frac{\partial \psi}{\partial \bar{z}}$ as $\sum_{j=1}^n \frac{\partial \psi}{\partial \bar{z}_j} d\bar{z}_j$. Now the norm of $\frac{\partial Y}{\partial \bar{z}}$ and $\frac{\partial \psi}{\partial \bar{z}}$ in terms of the metric h are:

$$\left| \frac{\partial Y}{\partial \bar{z}} \right|_h^2 = \sum_{\alpha, \beta, r, \delta=1}^n h_{\alpha\bar{r}} h^{\bar{\beta}\delta} \frac{\partial Y^\alpha}{\partial \bar{z}_\beta} \overline{\left(\frac{\partial Y^r}{\partial \bar{z}_\delta} \right)} \quad (22)$$

and

$$\left| \frac{\partial \psi}{\partial \bar{z}} \right|_h^2 = \sum_{\alpha, \beta=1}^n h^{\alpha\bar{\beta}} \frac{\partial \psi}{\partial z_\alpha} \frac{\partial \psi}{\partial \bar{z}_\beta}. \quad (23)$$

We claim the following inequality holds (for some uniform constant C):

$$\left| \frac{\partial X}{\partial \bar{z}} \right|_h \leq \sqrt{\sum_{\alpha, \beta, r, \delta=1}^n h_{\alpha\bar{r}} h^{\bar{\beta}\delta} \frac{\partial X^\alpha}{\partial \bar{z}_\beta} \overline{\left(\frac{\partial X^r}{\partial \bar{z}_\delta} \right)}} \leq C \sqrt{\text{tr}_g(h)} |D\varphi'|_g. \quad (24)$$

This could be proven by choosing a preferred coordinate, where $h_{i\bar{j}} = \delta_{i\bar{j}}$ ($1 \leq i, j \leq n$) while $g_{i\bar{j}} = \lambda_i \delta_{i\bar{j}}$ ($1 \leq i, j \leq n$) in an arbitrary point O . Here λ_i are eigenvalues of metric g in terms of metric h . These λ_i 's are uniformly bounded from above since g is uniformly bounded from above. We want to verify the above inequality in this point O .

$$\begin{aligned} \left| \frac{\partial X}{\partial \bar{z}} \right|_h^2 &= \sum_{\alpha, \beta, a, b=1}^n \frac{\partial X^\alpha}{\partial \bar{z}_\beta} \frac{\partial X^{\bar{a}}}{\partial z_b} h_{\alpha\bar{a}} h^{\bar{\beta}b} \\ &= \sum_{\alpha, \beta, a, b, c, d=1}^n h_{\alpha\bar{a}} h^{\bar{\beta}b} g^{\alpha\bar{c}} \varphi'_{, \bar{c}\bar{\beta}} \varphi'_{, d\bar{b}} g^{\bar{a}d} = \sum_{\alpha, \beta=1}^n \delta_{\alpha\bar{a}} \delta^{\bar{\beta}b} \frac{1}{\lambda_\alpha} \delta^{\alpha\bar{c}} \varphi'_{, \bar{c}\bar{\beta}} \varphi'_{, d\bar{b}} \frac{1}{\lambda_a} \delta^{\bar{a}d} \\ &= \sum_{\alpha, \beta=1}^n \frac{1}{\lambda_\alpha^2} \varphi'_{, \bar{\alpha}\bar{\beta}} \varphi'_{, \alpha\beta} \leq \left(\sum_{\alpha, \beta=1}^n \frac{\lambda_\beta}{\lambda_\alpha} \right) \sum_{\alpha, \beta=1}^n \frac{1}{\lambda_\alpha \lambda_\beta} \varphi'_{, \bar{\alpha}\bar{\beta}} \varphi'_{, \alpha\beta} \\ &\leq C \cdot \text{tr}_g(h) \cdot |D\varphi'|_g^2. \end{aligned}$$

Here C is a uniform constant. From inequality (21) and the fact g is bounded from above, we have

$$\int_{V \times I} \left| \nabla \log \frac{\det g}{\det h} \right|_h^2 = \int_{V \times I} |\nabla \log f|_h^2 \leq C \cdot \int_{V \times I} |\nabla \log f|_g^2 \det g dt \leq C.$$

and

$$\int_{V \times I} \left(\frac{\det h}{\det g} \right)^{\frac{1}{n}} \det h \leq \int_{V \times I} \text{tr}_g(h) \det h \leq C.$$

From now on, all of the norm, inner product and integration are taken w.r.t. metric $h + dt^2$ unless otherwise specified. Now define a new vector field Y as

$$Y = X \cdot \frac{\det g}{\det h}.$$

Then

$$|Y|_h = |X|_h \frac{\det g}{\det h} \leq C.$$

In other words, Y has uniform L^∞ bound. This implies that $Y \cdot \frac{\partial \ln \frac{\det g}{\det h}}{\partial \bar{z}}$ has uniform L^q bound for any $1 < q < 2$. Moreover, for any $1 < q < 2$, we have

$$\begin{aligned} \int_{V \times I} \left| \frac{\partial Y}{\partial \bar{z}} - Y \cdot \frac{\partial \ln \frac{\det g}{\det h}}{\partial \bar{z}} \right|_h^q &= \int_{V \times I} \left(\left| \frac{\partial X}{\partial \bar{z}} \right|_h \frac{\det g}{\det h} \right)^q \\ &\leq \int_{V \times I} (\sqrt{\text{tr}_g(h)} \frac{\det g}{\det h})^q |D\varphi'|_g^q \\ &\leq \int_{V \times I} C |D\varphi'|_g^q \rightarrow 0. \end{aligned}$$

This immediately implies that $\frac{\partial Y}{\partial \bar{z}}$ are uniformly bounded in L^q for any $1 < q < 2$.

Now, all of these quantities, $X, Y, \frac{\partial Y}{\partial \bar{z}}$, and $\frac{\det g}{\det h}, \dots$ are geometrical quantities which depend on ϵ . Since their respective soblev norms are uniformly controlled, we can take weak limits of these quantities in some appropriate sense. Denote the corresponding weak limits (when $\epsilon \rightarrow 0$) as $X, Y, \frac{\det g}{\det h}, \dots$. Then $X(\epsilon) \rightharpoonup X$ weakly in $L^2(V \times I)$, $Y(\epsilon) \rightharpoonup Y$ weakly in $L^\infty(V \times I)$ and $\frac{\det g}{\det h}(\epsilon) \rightharpoonup \frac{\det g}{\det h}$ weakly in $L^\infty(V \times I), \dots$.

Consider $u = \ln \frac{\det h}{\det g}$. For simplicity, assume $u > 0$ (otherwise $u > -c$ for some positive constants). Then the following two equations holds in the limit

$$\frac{\partial Y}{\partial \bar{z}} + Y \cdot \frac{\partial u}{\partial \bar{z}} = 0, \quad \text{and} \quad Y = X e^{-u}$$

in the sense of $L^q(V \times I)$ for any $1 < q < 2$. Moreover, we have the following estimates:

$$\int_{V \times I} e^{\frac{1}{n}u} \leq C; \quad \int_{V \times I} \left| \frac{\partial u}{\partial \bar{z}} \right|^2 \leq C; \quad \text{and} \quad \int_{V \times I} |X|^2 \leq C.$$

Now define a new sequence of vectors $X_{,k} (k = 1, 2, \dots)$ as $X_{,k} = Y \sum_{i=0}^k \frac{u^i}{i!}$. This is well defined since u is in $L^p(V \times I)$ for any $p > 1$. Then

$$\begin{aligned} |X_{,k}| &= |Y| \sum_{i=0}^k \frac{u^i}{i!} \\ &\leq (|X| e^{-u}) e^u \leq |X|. \end{aligned}$$

The equality holds in the last inequality whenever $e^{-u} \neq 0$. Thus

$$\int_{V \times I} |X_{,k}|^2 \leq \int_{V \times I} |X|^2 \leq C.$$

By definition, it is clear $\|X_{,k}\|_{L^2(V \times I)} \leq \|X_{,m}\|_{L^2(V \times I)}$ whenever $k \leq m$. Thus, there exists a positive number $A \leq \|X\|_{L^2(V \times I)}$ such that $\lim_{k \rightarrow \infty} \|X_{,k}\|_{L^2(V \times I)} = A$. For $m > k$, we have

$$\begin{aligned} \|X_{,m}\|_{L^2(V \times I)}^2 &= \int_{V \times I} |X_{,m}|^2 \\ &= \int_{V \times I} |Y|^2 \left(\sum_{i=0}^m \frac{u^i}{i!} \right)^2 \geq \int_{V \times I} |Y|^2 \left(\left(\sum_{i=0}^k \frac{u^i}{i!} \right)^2 + \left(\sum_{i=k+1}^m \frac{u^i}{i!} \right)^2 \right) \\ &= \int_{V \times I} (|X_{,k}|^2 + |X_{,m} - X_{,k}|^2) = \|X_{,k}\|_{L^2(V \times I)}^2 + \|X_{,m} - X_{,k}\|_{L^2(V \times I)}^2. \end{aligned}$$

Taking limits as $m, k \rightarrow \infty$, we have $\|X_{,m} - X_{,k}\|_{L^2(V \times I)}^2 \rightarrow 0$. Thus $X_{,k} (k = 1, 2, \dots)$ is a Cauchy sequence in $L^2(V \times I)$ and there exists a strong limit X_∞ in $L^2(V \times I)$. By definition, we know

that $X_\infty = X$ almost everywhere in $V \times I$ ^{††}. We want to show that X_∞ is weakly holomorphic in V -direction. A straightforward calculation yields

$$\begin{aligned} \frac{\partial X_k}{\partial \bar{z}} &= \frac{\partial Y}{\partial \bar{z}} \sum_{i=0}^k \frac{u^i}{i!} + Y \frac{\partial}{\partial \bar{z}} \left(\sum_{i=0}^k \frac{u^i}{i!} \right) \\ &= -Y \frac{\partial u}{\partial \bar{z}} \sum_{i=0}^k \frac{u^i}{i!} + Y \left(\sum_{i=0}^{k-1} \frac{u^i}{i!} \right) \frac{\partial u}{\partial \bar{z}} \\ &= -(X_{k,k} - X_{k,k-1}) \frac{\partial u}{\partial \bar{z}}. \end{aligned}$$

We want to show that $\frac{\partial X_\infty}{\partial \bar{z}} = 0$ in the sense of distribution. We just need to show it in any open set $U \times I$ where U is a coordinate chart in V . Denote (z_1, z_2, \dots, z_n) as coordinate variable in U . Then, for any vector valued smooth function $\psi = (\psi^1, \psi^2, \dots, \psi^n)$ which vanish in $\partial(U \times I)$, and for any $1 \leq j \leq n$, we have

$$\begin{aligned} \left| \int_{V \times I} X_{k,k} \cdot \frac{\partial \bar{\psi}}{\partial \bar{z}_j} \right| &= \left| - \int_{V \times I} \frac{\partial X_{k,k}}{\partial \bar{z}_j} \cdot \bar{\psi} \right| \\ &= \left| \int_{V \times I} (X_{k,k} - X_{k,k-1}) \frac{\partial u}{\partial \bar{z}_j} \bar{\psi} \right| \\ &\leq C \cdot \|X_{k,k} - X_{k,k-1}\|_{L^2(V \times I)} \cdot \sqrt{\int_{V \times I} |\nabla u|^2} \\ &\leq C \|X_{k,k} - X_{k,k-1}\|_{L^2(V \times I)}. \end{aligned}$$

Now, taking limit as $k \rightarrow \infty$, we have

$$\int_{V \times I} X_\infty \cdot \frac{\partial \bar{\psi}}{\partial \bar{z}_j} = 0, \quad \text{for any } j = 1, 2, \dots, n$$

and for any smooth vector valued function $\psi = (\psi^1, \psi^2, \dots, \psi^n)$ which vanish in $\partial(U \times I)$. Thus, X_∞ is a weak holomorphic vector field in V direction for almost all t .

Now recalls that

$$\int_{V \times I} |X_\infty|_h^2 \det h \, dt < C.$$

This implies that X_∞ is in $L^2(V \times \{t\})$ for almost all $t \in [0, 1]$. Since X_∞ is weakly holomorphic in $V \times \{t\}$ for all t , thus X_∞ must be holomorphic for those t where X_∞ is in $L^2(V \times \{t\})$. However, there is no holomorphic vector field in V since $C_1 < 0$. Thus $X_\infty \equiv 0$ for all of those t where X_∞ is in $L^2(V \times \{t\})$. This implies that $X_\infty = 0$ in $V \times I$. Thus $X = 0$ since $X = X_\infty$ in the sense of $L^q(V \times I)$ for any $1 < q < 2$. Recall

$$\frac{\partial \varphi'(t)}{\partial z_\alpha} = \sum_{\beta=1}^n g_{\alpha\bar{\beta}} X^\beta = \sum_{\beta=1}^n g_{\alpha\bar{\beta}} X_{,\infty}^\beta = 0.$$

In other words, $\varphi'(t)$ is trivial in V -direction and it is a function of t only for all $t \in [0, 1]$. Thus, $\varphi(0)$ and $\varphi(1)$ differ only by a constant in V direction. Therefore they represent same metric in each Kähler class.

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^{††}It is easy to prove that $X_\infty = X$ in the sense of $L^q(V \times I)$ for any $(1 < q < 2)$.

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